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SOME RESULTS ON THE NONSTATIONARY IDEAL

BY

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ABSTRACT

The strength of precipitousness, presaturatedness and saturatedness of NS_κ and $\mathrm{NS}_\kappa^\lambda$ is studied. In particular, it is shown that:

(1) The exact strength of "$\mathrm{NS}_{\mu^+}^\lambda$ for a regular $\mu > \max(\lambda, \aleph_1)$" is a (ω, μ)-repeat point.

(2) The exact strength of "NS_κ is presaturated over inaccessible κ" is an up-repeat point.

(3) "NS_κ is saturated over inaccessible κ" implies an inner model with $\exists\alpha o(\alpha) = \alpha^{++}$.

Introduction

The strength of the following basic hypothesis on NS_κ (the nonstationary ideal over κ) and $\mathrm{NS}_\kappa^\lambda$ (the nonstationary ideal over κ restricted to the cofinality λ) will be studied:

1. Precipitousness (i.e., the generic ultrapower is well founded).
2. Presaturatedness (i.e., precipitousness + all the cardinals except perhaps κ itself are preserved in the forcing extension by NS_κ).
3. Saturatedness (i.e., the forcing with NS_κ satisfies κ^+-c.c.).

1. PRECIPITOUSNESS. By Jech–Magidor–Mitchell–Prikry, "$\mathrm{NS}_{\aleph_1}$ precipitous" or "$\mathrm{NS}_{\kappa^+}^\kappa$ precipitous for regular κ" is equiconsistent with a measurable. By [G2], "$\mathrm{NS}_{\aleph_2}^{\aleph_0}$ precipitous" is equiconsistent with a measurable and "$\mathrm{NS}_{\aleph_2}$" is equiconsistent with a measurable of order 2.

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Models with NS_κ precipitous for $\kappa > \aleph_2$ were constructed in [G3] and [F-M-S] from strong assumptions. T. Jech [J2] gives a lower bound on the strength of "NS_κ precipitous".

For $X \subseteq \kappa$ denote by $\mathrm{NS}_\kappa \restriction X$ the nonstationary ideal restricted to X, i.e. the set of all $A \subseteq \kappa$ such that $A \cap X$ is nonstationary. The following will be proved here:

(a) The strength of "$\mathrm{NS}_\kappa^\lambda$ or $\mathrm{NS}_\kappa \restriction$(Singular cardinals) is precipitous for an inaccessible κ, $\lambda < \kappa$" is at least an $(\omega, <\kappa)$-repeat point.

(b) The strength of $\mathrm{NS}_\kappa^{\aleph_0}$ precipitous, $\kappa = \mu^{++}$, $\mathrm{cf}\,\mu = \omega$ + GCH is at least an $(\omega, <\mu)$-repeat point.

(c) The exact strength of "($\mathrm{NS}_\kappa^\lambda$ or $\mathrm{NS}_\kappa \mid \{\alpha < \kappa \mid \mathrm{cf}\ \alpha < \mu\}$ is precipitous) $+\kappa = \mu^+$ for a regular $\mu > \max(\lambda, \aleph_1)$ + GCH" is an (ω, μ)-repeat point.

(d) The existence of an $(\omega, \kappa^+ + 1)$-repeat point is sufficient for "NS_κ precipitous $+\kappa$ is an inaccessible + GCH".

(e) The existence of an $(\omega, \mu + 1)$-repeat point is sufficient for "NS_κ is precipitous $+\kappa = \mu^+$ + GCH".

2. AND 3. PRESATURATEDNESS, SATURATEDNESS. By S. Shelah [S], $\mathrm{NS}_{\kappa^+}^\lambda$ cannot be presaturated for $\lambda < \kappa$. Models with $\mathrm{NS}_{\aleph_1}$ saturated were constructed by J. Steel and R. Van Wesep [S-V], H. Woodin [W] using AD and M. Foreman, M. Magidor and S. Shelah [F-M-S] from a supercompact cardinal. By W. Mitchell [Mi2], presaturatedness of $\mathrm{NS}_{\kappa^+}^\kappa$ implies an inner model with $\exists\kappa\, O(\kappa) = \kappa^{++}$. For inaccessible κ, T. Jech and H. Woodin [J-W] showed that $\mathrm{NS}_\kappa \restriction$(regular cardinals) can be saturated from one measurable and by [G3] $\mathrm{NS}_\kappa \restriction S$ can be saturated for a set S stationary in every cofinality $< \kappa$, from $O(\kappa) = \kappa$. It will be shown here that

(a) The exact strength of "$\mathrm{NS}_\kappa^\lambda$ presaturated for $\lambda < \kappa$ over inaccessible κ" is an up-repeat point.

(b) The exact strength of "NS_κ presaturated over inaccessible κ" is an up-repeat point.

(c) "NS_κ is saturated for inaccessible κ" implies an inner model with $\exists\kappa\, O(\kappa) = \kappa^{++}$.

The paper is organized as follows. In Section 1 various notions of repeat points are introduced. Using the Core model techniques, lower bounds on the strength of the existence of the ideals are found in Section 2. Precipitous ideals are constructed in Section 3 and the presaturated ideals in Section 4. The situation over a measurable cardinal is studied in Section 5. A knowledge of Mitchell's Covering Lemma [Mi3,4] is required for Section 2. Sections 3 and 4 can be read independently of Section 2, but we assume there a familiarity with forcings of [G2,3,4]. For most notation and basic definitions we refer to T. Jech [J1] and A. Kanamori and M. Magidor [K-M].

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1. Repeat points

The notion of repeat point was originally introduced by Radin [R] in order to preserve the measurability under the Radin forcing. The existence of his repeat point required a $\mathcal{P}^3(\kappa)$-measurable cardinal. Mitchell [Mi1] used a weaker notion which does not require even $O(\kappa) = \kappa^{++}$. Let us define some intermediate repeat points which will be used in Sections 2, 3 and 4.

Let $\vec{\mathcal{F}}$ be a coherent sequence of ultrafilters of the length $\ell^{\vec{\mathcal{F}}}$.

Definition 1.1: Let κ be a cardinal $< \ell^{\vec{\mathcal{F}}}$ and $\alpha, \delta < O^{\vec{\mathcal{F}}}(\kappa)$, $\delta > 0$.

(1) (Mitchell [Mi1]) α is a **weak repeat point** for $\vec{\mathcal{F}}$ at κ if for each set A in $\mathcal{F}(\kappa, \alpha)$ there is $\beta < \alpha$ such that $A \in \mathcal{F}(\kappa, \beta)$.

(2) α is a δ-**repeat point** if for each set A in $\bigcap\{\mathcal{F}(\kappa,\gamma)\colon \alpha \leq \gamma < \alpha+\delta\}$ there is $\beta < \alpha$ such that $A \in \cap\{\mathcal{F}(\kappa,\gamma)\colon \beta \leq \gamma < \beta+\delta\}$.

(3) α is a $< \delta$-**repeat point** if α is a δ'-repeat point for every $\delta' < \delta$.

(4) α is an **up-repeat** point if for each set A in $\mathcal{F}(\kappa, \alpha)$ there is $\beta > \alpha$ such that $A \in \mathcal{F}(\kappa, \beta)$.

We will be interested in δ-repeat points for relatively small δ's like $\delta = \omega_2$, $\delta = \kappa^+$, $\delta = \kappa^+ + 1$ etc. These δ's can be represented by the same function in all the ultrapowers of $\mathcal{K}(\mathcal{F})$ with $\mathcal{F}(\kappa,\gamma)$ for $\gamma < O^{\vec{\mathcal{F}}}(\kappa)$. Let us call ordinals with this property **uniformly representable**.

LEMMA 1.2: *Suppose that α is a ξ-repeat point. Then, for every uniformly representable $\eta \leq \xi$, α is an η-repeat point.*

Proof: Suppose that $\eta < \xi$. Let g be a function which uniformly represents η.

Let $A \in \bigcap\{\mathcal{F}(\kappa,\gamma) \mid \alpha \le \gamma < \alpha+\eta\}$. Consider a set $B = \{\delta < \kappa \mid$ there is $\delta^* < O^{\vec{\mathcal{F}}}(\delta)$ such that $\delta^* + g(\delta) \le O^{\vec{\mathcal{F}}}(\delta)$ and $A \bigcap \delta \in \cap\{\mathcal{F}(\delta,\gamma) \mid \delta^* \le \gamma < \delta^* + g(\delta)\}\}$.

The set B belongs to every measure $\mathcal{F}(\kappa,\rho)$ with $\rho \ge \alpha+\eta$. Hence there is $\beta < \alpha$ such that

$$A \cup B \in \cap\{\mathcal{F}(\kappa,\gamma) \mid \beta \le \gamma < \beta+\xi\}\ .$$

If $A \in \bigcap\{\mathcal{F}(\kappa,\gamma) \mid \beta \le \gamma < \beta+\xi\}$, then we are done. Otherwise for $\bar{\beta}$, $\beta \le \bar{\beta} < \beta+\xi$, $B \in \mathcal{F}(\kappa,\bar{\beta})$. But then there is $\beta^* < \bar{\beta}$ so that $\beta^*+\eta \le \bar{\beta}$ and $A \in \bigcap\{\mathcal{F}(\kappa,\gamma) \mid \beta^* \le \gamma < \beta^*+\eta\}$. ∎

In particular, if α is an η-repeat point then α is a weak repeat point.

The proof of Lemma 1.2 gives a little more.

Lemma 1.3: *Let $\alpha < \xi$ be ordinals. Suppose that for every $A \in \mathcal{F}(\kappa,\xi)$ there is $\beta < \alpha$ such that $A \in \mathcal{F}(\kappa,\beta)$. Let η be uniformly representable and $\alpha+\eta \le \xi$. Then α is an η-repeat point.*

Lemma 1.4: *If α is an up-repeat point then α is a δ-repeat point for every uniformly representable $\delta < \alpha$.*

Proof: Pick $g_\delta\colon \kappa \to \kappa$ which uniformly represents δ. Suppose that $A \in \bigcap\{\mathcal{F}(\kappa,\gamma) \mid \alpha \le \gamma < \alpha+\delta\}$ and for each $\beta < \alpha$ we have $A \notin \bigcap\{\mathcal{F}(\kappa,\gamma) \mid \beta \le \gamma < \beta+\delta\}$. Define a set $B = \{\tau < \kappa \mid$ for every $\xi < O^{\vec{\mathcal{F}}}(\tau)$ we have $A \cap \tau \notin \bigcap\{\mathcal{F}(\tau,\gamma) \mid \xi \le \gamma < \xi + g_\delta(\tau)\}\}$.

Then $B \in \mathcal{F}(\kappa,\alpha)$. So for some $\eta > \alpha$, $B \in \mathcal{F}(\kappa,\eta)$. But then, in the ultrapower with $\mathcal{F}(\kappa,\eta)$, $A \notin \bigcap\{\mathcal{F}(\kappa,\gamma) \mid \alpha \le \gamma < \alpha+\delta\}$ which is impossible. Contradiction. ∎

Actually, we shall see that the presence of an up-repeat point implies a lot of repeat points.

The following lemma is clear

Lemma 1.5: *Suppose that $2^\kappa = \kappa^+$ and $O^{\vec{\mathcal{F}}}(\kappa) = \kappa^{++}$. Then every sufficiently large $\alpha < \kappa^{++}$ is an up-repeat point and in fact every sufficiently large $\alpha < \kappa^{++}$ has the property that for every set $A \in \mathcal{F}(\kappa,\alpha)$, A belongs to $\mathcal{F}(\kappa,\beta)$ for unboundedly many β's below κ^{++}.*

Lemma 1.6: *Let α be the least up-repeat point. Then α is a limit ordinal and $\operatorname{cf} \alpha \ge \kappa^+$.*

Proof: Let us show first that α is a limit ordinal. Suppose otherwise. Let $\alpha = \alpha^* + 1$ and $B \in \mathcal{F}(\kappa, \alpha^*)$. Then $A = \{\tau < \kappa \mid O^{\vec{\mathcal{F}}}(\tau) = \tau^* + 1$ for some τ^* and $B \cap \tau \in \mathcal{F}(\tau, \tau^*)\}$ is in $\mathcal{F}(\kappa, \alpha)$. There exists $\beta > \alpha$ such that $A \in \mathcal{F}(\kappa, \beta)$. So, $B \in \mathcal{F}(\kappa, \beta - 1)$ and $\beta - 1 > \alpha^*$. Hence α^* is an up-repeat point; which is impossible.

So α is a limit ordinal. Suppose that cf $\alpha = \lambda < \kappa^+$. Pick a cofinal sequence $\langle c_i \mid i < \lambda \rangle$ to α. For every $i < \lambda$ let $A_i \in \mathcal{F}(\kappa, c_i)$ and $A_i \notin \mathcal{F}(\kappa, \beta)$ for $\beta > c_i$.

Let $A = \{\tau < \kappa \mid$ (a) $O^{\mathcal{F}}(\tau)$ is a limit ordinal of cofinality λ, if $\lambda < \kappa$ or of cofinality τ, if $\lambda = \kappa$ and (b) there exists a cofinal sequence $\langle \tau_i \mid i < \mathrm{cf}\ O^{\vec{\mathcal{F}}}(\tau) \rangle$ to $O^{\mathcal{F}}(\tau)$ so that $A_i \cap \tau \in \mathcal{F}(\tau, \tau_i)$ for every $i < \mathrm{cf}\ O^{\vec{\mathcal{F}}}(\tau)\}$. Then $A \in \mathcal{F}(\kappa, \alpha)$. Hence, for some $\beta > \alpha$, $A \in \mathcal{F}(\kappa, \beta)$. Then in the ultrapower with $\mathcal{F}(\kappa, \beta)$, $A_i \in \mathcal{F}(\kappa, \tau_i)$ $(i < \lambda)$ and $\langle \tau_i \mid i < \lambda \rangle$ is cofinal in β. So for some i_0 every $i > i_0$ is above α. But $A_i \notin \mathcal{F}(\kappa, \gamma)$ for $\gamma > \gamma_i$ and $\gamma_i < \alpha$. Contradiction. ∎

LEMMA 1.7: *Suppose that $2^\kappa = \kappa^+$, α is a weak repeat point and there is no κ^+-repeat point below α. Then either*

(1) cf $\alpha = \kappa^+$

or

(2) *there is $\alpha' < \alpha$ such that* cf $\alpha' = \kappa^+$ *and $\alpha < \alpha' + \kappa^+$.*

Proof: Suppose otherwise. Let $\alpha^* \leq \alpha$ be the minimal ordinal such that $\alpha^* + \kappa^+ > \alpha$ and $\beta + \beta' < \alpha^*$ for every $\beta < \alpha^*$ and $\beta' < \kappa^+$. Pick a cofinal sequence $\langle \beta_i \mid i < \mathrm{cf}\ \alpha^* \rangle$ to α^*. Let $\langle A_\nu \mid \nu < \kappa^+ \rangle$ be an almost decreasing generating family for $\mathcal{F}(\kappa, \alpha)$. Since $\alpha - \alpha^* < \kappa^+$ and $\mathcal{F}(\kappa, \gamma_1) \neq \mathcal{F}(\kappa, \gamma_2)$ for $\gamma_1 \neq \gamma_2$, there exists $i_0 < \mathrm{cf}\ \alpha^*$ so that κ^+ of A_ν's belong to measures below β_{i_0}. But then every $A \in \mathcal{F}(\kappa, \alpha)$ belongs to some $\mathcal{F}(\kappa, \beta)$ with $\beta < \beta_{i_0}$. Hence by Lemma 1.3, β_{i_0} is κ^+-repeat point. Contradiction. ∎

LEMMA 1.8: *Let α be an up-repeat point for $\vec{\mathcal{F}}$ at κ. Suppose that $\bar{\alpha} \leq O^{\vec{\mathcal{F}}}(\kappa)$ is minimal so that α is an up-repeat point for $\vec{\mathcal{F}} \mid (\kappa, \bar{\alpha})$. Then for every β, $\alpha \leq \beta < \bar{\alpha}$, every $A \in \mathcal{F}(\kappa, \beta)$ there is $\gamma < \alpha$ such that $A \in \mathcal{F}(\kappa, \gamma)$.*

Proof: Suppose otherwise. Then there is β, $\alpha \leq \beta < \bar{\alpha}$, $A \in \mathcal{F}(\kappa, \beta)$ so that $A \notin \mathcal{F}(\kappa, \gamma)$ for every $\gamma < \alpha$. Let

$$B = \{\tau < \kappa \mid \text{for every } \tau' < O^{\vec{\mathcal{F}}}(\tau) \text{ we have } A \cap \tau \notin \mathcal{F}(\tau, \tau')\}.$$

Then $B \in \mathcal{F}(\kappa, \alpha)$. So there is α^* such that $\beta < \alpha^* < \bar{\alpha}$ and $B \in \mathcal{F}(\kappa, \alpha^*)$.

Hence, in the ultrapower with $\mathcal{F}(\kappa,\alpha^*)$, $A \notin \mathcal{F}(\kappa,\tau')$ for every $\tau' < \alpha^*$, which is impossible, since $A \in \mathcal{F}(\kappa,\beta)$ and $\beta < \alpha^*$. ∎

The lemma implies that every β with $\alpha \le \beta < \bar{\alpha}$ is a weak repeat point. Let us show more.

PROPOSITION 1.9: *Let α be the least up-repeat point for $\vec{\mathcal{F}}$ at κ. Suppose that $\bar{\alpha} \le O^{\vec{\mathcal{F}}}(\kappa)$ is minimal so that α is an up-repeat point for $\vec{\mathcal{F}} \mid (\kappa,\bar{\alpha})$. Then every β with $\alpha \le \beta < \bar{\alpha}$ is an up-repeat point for $\vec{\mathcal{F}}$ at κ.*

Proof: Let $\alpha < \beta < \bar{\alpha}$ and $B \in \mathcal{F}(\kappa,\beta)$. Set

$$D = \{\delta < \alpha \mid B \in \mathcal{F}(\kappa,\delta)\}.$$

By Lemma 1.6, $D \neq \emptyset$. Let us split the proof into two cases:

(1) D is unbounded in α,

(2) D is bounded in α.

CASE 1: *D is unbounded in α.* Then the set $A = \{\tau < \kappa \mid$ for every $\tau' < O^{\vec{\mathcal{F}}}(\tau)$ there exists τ'' such that $\tau < \tau'' < O^{\vec{\mathcal{F}}}(\tau)$ and $B \cap \tau \in \mathcal{F}(\tau,\tau'')\}$ is in $\mathcal{F}(\kappa,\alpha)$. So there is $\gamma > \beta + 1$ such that $A \in \mathcal{F}(\kappa,\gamma)$. Then the following holds in the ultrapower with $\mathcal{F}(\kappa,\gamma)$. For every $\tau' < \gamma$ there is τ'' with $\gamma > \tau'' > \tau'$ such that $B \in \mathcal{F}(\kappa,\tau'')$. Hence there is $\tau'' > \beta$ such that $B \in \mathcal{F}(\kappa,\tau'')$.

CASE 2: *D is bounded in α.* Let $\delta = \bigcup D$. Let $A_\delta \in \mathcal{F}(\kappa,\delta)$ be such that $A_\delta \notin \mathcal{F}(\kappa,\gamma)$ for every $\gamma > \delta$. Set $A = \{\tau < \kappa \mid$ there exists a maximal $\bar{\delta} < O^{\vec{\mathcal{F}}}(\tau)$ s.t. $A_\delta \cap \tau \in \mathcal{F}(\tau,\bar{\delta})$ and $B \cap \tau \notin \mathcal{F}(\tau,\tau')$ for every τ' , $\bar{\delta} < \tau < O^{\vec{\mathcal{F}}}(\tau)\}$.

Then $A \in \mathcal{F}(\kappa,\alpha)$. So for some $\gamma > \beta$, $A \in \mathcal{F}(\kappa,\gamma)$. There exists a maximal $\bar{\delta} < \gamma$ such that $A_\delta \in \mathcal{F}(\kappa,\bar{\delta})$ and $B \notin \mathcal{F}(\kappa,\tau')$ for every τ' with $\bar{\delta} < \tau' < \gamma$. But by the choice of A_δ, $\delta = \bar{\delta}$. On the other hand $\gamma > \beta > \delta$ and $B \in \mathcal{F}(\kappa,\beta)$. Contradiction. ∎

The following follows from the proof of Proposition 1.9.

COROLLARY 1.10: *Let $\alpha, \bar{\alpha}$ be as in 1.9. Then for every β with $\alpha \le \beta < \bar{\alpha}$ and for every $B \in \mathcal{F}(\kappa,\beta)$ there are unboundedly many δ's below α such that $B \in \mathcal{F}(\kappa,\delta)$.*

PROPOSITION 1.11: *Let $\alpha, \bar{\alpha}$ be as in 1.9. Define $\mathcal{F}_0$ to be a set of all $A \subseteq \kappa$ so that, for some $\gamma < \alpha$, $A \in \bigcap_{\alpha \ge \delta > \gamma} \mathcal{F}(\kappa,\delta)$ and let $\mathcal{F}_1$ be a set of all $A \subseteq \kappa$ so that, for some $\gamma < \bar{\alpha}$, $A \in \bigcap_{\gamma < \delta < \bar{\alpha}} \mathcal{F}(\kappa,\delta)$. Then $\mathcal{F}_0 = \mathcal{F}_1$.*

Proof: By Corollary 1.10, $\mathcal{F}_0 \subseteq \mathcal{F}_1$. Let us show the opposite direction. Let $A \in \mathcal{F}_1$. Suppose $A \notin \mathcal{F}_0$. Then for every $\delta < \alpha$ there is δ' such that $\delta < \delta' < \alpha$ and $A \notin \mathcal{F}(\kappa, \delta')$. Set $B = \{\tau < \kappa \mid$ for every $\delta < O^{\vec{\mathcal{F}}}(\tau)$ there is δ', $\delta < \delta' < O^{\vec{\mathcal{F}}}(\tau)$ s.t. $A \cap \tau \notin \mathcal{F}(\tau, \delta')\}$.

Then $B \in \mathcal{F}(\kappa, \alpha)$. Pick $\delta < \bar{\alpha}$ so that $A \in \mathcal{F}(\kappa, \gamma)$ for every γ, $\delta \leq \gamma < \bar{\alpha}$. For some δ^* and $\delta < \delta^* < \bar{\alpha}$, $B \in \mathcal{F}(\kappa, \delta^*)$. But then for some δ' with $\delta \leq \delta' < \delta^*$, $A \notin \mathcal{F}(\kappa, \delta')$. Contradiction. ∎

Note that $\mathcal{F}_0$ is a normal filter over κ. The property above looks similar to the Radin repeating measure which is stronger than $O(\kappa) = \kappa^{++}$.

Proposition 1.12: *Let $\alpha < \bar{\alpha} \leq O^{\vec{\mathcal{F}}}(\kappa)$ be ordinals with* cf $\alpha > \omega$. *Define $\mathcal{F}_0$, $\mathcal{F}_1$ as in 1.11. If $\mathcal{F}_0 = \mathcal{F}_1$ and there is no up-repeat point below α, then α is an up-repeat point.*

Proof: Suppose otherwise. Let $A_\alpha \in \mathcal{F}(\kappa, \alpha)$ not belong to any $\mathcal{F}(\kappa, \beta)$ with $\beta > \alpha$. Let $D = \{\xi < \alpha \mid A_\alpha \in \mathcal{F}(\kappa, \xi)\}$ and $\delta = \bigcup D$.

Suppose that $\delta < \alpha$. Pick a set $A_\delta \in \mathcal{F}(\kappa, \delta)$ and $A_\delta \notin \bigcup_{\mu > \delta} \mathcal{F}(\kappa, \mu)$. Set $E = \{\tau < \kappa \mid$ there is $\bar{\delta} < O^{\vec{\mathcal{F}}}(\tau)$ such that (i) $A_\delta \cap \tau \in \mathcal{F}(\tau, \bar{\delta})$, (ii) $A_\delta \cap \tau \notin \bigcup_{\mu > \bar{\delta}} \mathcal{F}(\tau, \mu)$, (iii) $A_\alpha \cap \tau \notin \bigcup_{\mu \geq \bar{\delta}} \mathcal{F}(\tau, \mu)\}$.

Then $E \in \mathcal{F}(\kappa, \gamma)$ for every γ with $\alpha \geq \gamma > \delta$. So $E \in \mathcal{F}_1$. Pick some $\gamma > \alpha$ s.t. $E \in \mathcal{F}(\kappa, \gamma)$. Then in the ultrapower with $\mathcal{F}(\kappa, \gamma)$, $\bar{\delta} = \delta$ but $A_\alpha \in \mathcal{F}(\kappa, \alpha)$ and $\alpha < \gamma$. Contradiction. So $\delta = \alpha$. Note that A_α can be replaced by any $A \in \mathcal{F}(\kappa, \alpha)$. Hence α is a weak repeat point.

The set $A^* = \{\tau < \kappa \mid$ there exists the largest measure to which $A_\alpha \cap \tau$ belongs$\}$ is in $\mathcal{F}_1$. Since $\mathcal{F}_0 = \mathcal{F}_1$, there is $\gamma_0 < \alpha$ so that $A^* \in \bigcap_{\gamma_0 < \gamma < \alpha} \mathcal{F}(\kappa, \gamma)$. Then for every γ, $\gamma_0 < \gamma < \alpha$ there is $\gamma^* < \gamma$ s.t. $A_\alpha \in \mathcal{F}(\kappa, \gamma^*)$ and A_α does not belong to any $\mathcal{F}(\kappa, \gamma')$ with $\gamma^* < \gamma' < \gamma$. Define an increasing sequence $\langle \gamma_n \mid 0 < n < \omega \rangle$ of ordinals $> \gamma_0$ so that for every n, $\gamma_n^* < \gamma_{n+1}^*$. It is possible since $\delta = \alpha$. Let $\gamma_\omega = \bigcup_{n < \omega} \gamma_n$. Then $\gamma_\omega < \alpha$ and $A^* \notin \mathcal{F}(\kappa, \gamma_\omega)$. Contradiction. ∎

Proposition 1.13: *Let α be the least up-repeat point for $\vec{\mathcal{F}}$ at κ and let δ be uniformly representable. Then the set of β's below α such that*

(a) *β is a δ-repeat point;*
(b) *every set $A \in \bigcap\{\mathcal{F}(\kappa, \beta') \mid \beta \leq \beta' < \beta + \delta\}$ belongs to $\bigcap\{\mathcal{F}(\kappa, \gamma') \mid \gamma \leq \gamma' < \gamma + \delta\}$ for unboundedly many γ's in β,*

is a club in α.

Proof: Suppose otherwise. Let $S = \{\beta < \alpha \mid \beta$ does not satisfy (a) or (b)$\}$. Find stationary $S_1 \subseteq S$ and $\beta_0 < \alpha$ s.t. for every $\beta \in S_1$ there exists a set in $\bigcap\{\mathcal{F}(\kappa,\beta') \mid \beta \leq \beta' < \beta+\delta\}$ which does not belong to $\bigcap\{\mathcal{F}(\kappa,\gamma') \mid \gamma \leq \gamma' < \gamma+\delta\}$ for any γ, $\beta_0 < \gamma < \beta$. Pick a set $A_0 \in \mathcal{F}(\kappa,\beta_0)$ which does not belong to any measure above β_0. Consider a set $B = \{\rho < \kappa \mid$ there exists the maximal $\rho_0 < O^{\vec{\mathcal{F}}}(\rho)$ s.t. $A_0 \cap \rho_0 \in \mathcal{F}(\rho,\rho_0)$ and for unboundedly many β's in $O^{\vec{\mathcal{F}}}(\rho)$ there exists a set in $\bigcap\{\mathcal{F}(\rho,\rho') \mid \beta \leq \rho' < \beta + g_\delta(\rho)\}$ which does not belong to $\bigcap\{\mathcal{F}(\rho,\gamma') \mid \gamma \leq \gamma' < \gamma + g_\delta(\rho)\}$ for every γ, with $\delta_0 < \gamma < \beta\}$, where g_δ is a function uniformly representing δ. Clearly, $B \in \mathcal{F}(\kappa,\alpha)$. So $B \in \mathcal{F}(\kappa,\alpha')$ for some $\alpha' > \alpha$. Pick β, so that $\alpha \leq \beta < \alpha'$ and for some $A \in \cap\{\mathcal{F}(\kappa,\beta') \mid \beta \leq \beta' < \beta+\delta\}$ does not belong to $\bigcap\{\mathcal{F}(\kappa,\gamma') \mid \gamma \leq \gamma' < \gamma+\delta\}$ for every γ, with $\beta_0 < \gamma < \beta$. Now, as the proof of Proposition 1.9, we obtain the contradiction. ∎

Definition 1.14: Let α, δ be ordinals below $O^{\vec{\mathcal{F}}}(\kappa)$, $\delta > 0$ and let $\lambda \leq \kappa$ be a regular cardinal. Then α is a (λ,δ)-repeat point if (1) cf $\alpha = \lambda$, (2) α is a δ-repeat point such that every $A \in \bigcap\{\mathcal{F}(\kappa,\alpha') \mid \alpha \leq \alpha' < \alpha+\delta\}$ belongs to $\bigcap\{\mathcal{F}(\kappa,\gamma') \mid \gamma \leq \gamma' < \gamma+\delta\}$ for unboundedly many γ's in α. α is a $(\lambda, <\delta)$-repeat point if for every $\delta' < \delta$, α is a (λ,δ')-repeat point.

It follows by Lemma 1.6 and Proposition 1.13 that there are unboundedly many (λ,δ)-repeat points below the first up-repeat point for any regular $\lambda \leq \kappa$ and uniformly representable δ.

LEMMA 1.15: *Suppose that $\lambda_1 < \lambda_2 < \kappa^+$ are regular cardinals, δ is uniformly representable and α is a $(\lambda_2, 1)$-repeat point. If there is no up-repeat point, then there are unboundedly many (λ_1,δ)-repeat points below α.*

Proof: Choose an increasing unbounded in α sequence $\langle \alpha_i \mid i < \lambda_2\rangle$ such that for every $A \in \mathcal{F}(\kappa,\alpha)$, for every $i < \lambda_2$ there is $\beta \in (\alpha_i, \alpha_{i+1})$ so that $A \in \mathcal{F}(\kappa,\beta)$. For every $i < \lambda_2$ pick a set $A_i \in \mathcal{F}(\kappa,\alpha_i)$ which does not belong to any measure above α_i.

Let $\beta = \bigcup_{i<\lambda_1} \alpha_i$ and $A \in \bigcap\{\mathcal{F}(\kappa,\beta') \mid \beta \leq \beta' < \beta+\delta\}$. Fix some $i_0 < \lambda_1$. Consider the set $B = \{\rho < \kappa \mid$ there exists the maximal $\rho_0 < O^{\vec{\mathcal{F}}}(\rho)$ s.t. $A_{i_0} \cap \rho \in \mathcal{F}(\kappa,\rho_0)$; for some γ, $\gamma_0 < \gamma < O^{\vec{\mathcal{F}}}(\rho)$ $A \cap \delta \in \bigcap\{\mathcal{F}(\rho,\gamma') \mid \gamma \leq \gamma' < \gamma + g_\delta(\rho)\}\}$. Clearly $B \in \mathcal{F}(\kappa,\alpha)$. Then for some τ with $\alpha_{i_0} < \tau < \alpha_{i_0+1}$, $B \in \mathcal{F}(\kappa,\tau)$. By the choice of B, then there is γ, $\alpha_{i_0} < \gamma < \tau$ so that $A \in \bigcap\{\mathcal{F}(\kappa,\gamma) \mid \gamma \leq \gamma' < \gamma+\delta\}$. It means that β is a (λ_1,δ)-repeat point. ∎

LEMMA 1.16: *Suppose there is no up-repeat point. Let λ be a regular cardinal, α be a weak repeat point and $\langle \alpha_i \mid i < \lambda \rangle$ be an increasing sequence of ordinals below α so that for every $A \in \mathcal{F}(\kappa, \alpha)$, every $i < \lambda$, $A \in \bigcup\{\mathcal{F}(\kappa, \beta) \mid \alpha_i < \beta < \alpha_{i+1}\}$. Then $\bigcup_{i<\lambda} \alpha_i$ is a (λ, δ)-repeat point for every uniformly representable δ.*

Use the proof of Lemma 1.15.

2. The lower bounds

Let $\mathrm{NS}_\kappa^{\aleph_0}$ denote the ideal of all ω-nonstationary subsets of κ, i.e., all the subsets X of κ so that $X \cap \{\delta < \kappa \mid \mathrm{cf}\ \delta = \aleph_0\}$ is nonstationary.

In this section we are going to prove the following.

THEOREM 2.0: *Let κ be a regular cardinal above $\aleph_2$ and $\mathrm{NS}_\kappa^{\aleph_0}$ be precipitous. Then*

1. *If $\kappa = \lambda^{++}$ for λ of cofinality $> \omega$ and $\lambda^\omega = \lambda$, then there exists an (ω, λ^+)-repeat point.*
2. *If $\kappa = \lambda^{++}$ for λ of cofinality ω and $\mu^\omega < \lambda$ for every $\mu < \lambda$, then there exists an $(\omega, < \lambda)$-repeat point.*
3. *If $\kappa = \lambda^+$ for a weakly inaccessible λ and $\mu^\omega < \lambda$ for every $\mu < \lambda$, then there exists an (ω, λ)-repeat point.*
4. *If κ is a weakly inaccessible and $\mu^\omega < \kappa$ for every $\mu < \kappa$, then there exists an $(\omega, < \kappa)$-repeat point.*
5. *If after the forcing with $\mathrm{NS}_\kappa^{\aleph_0}$, $\kappa \geq ((2^\omega)^+)$, then there exists an up-repeat point.*
6. *If NS_κ is κ^+-saturated, then $(\exists \alpha\ O(\alpha) = \alpha^{++})$.*

Note that if $\kappa = \lambda^+$ for a singular λ, then by [Mi2] $\exists \alpha\ O(\alpha) = \alpha^{++}$ since the generic ultrapower with $\mathrm{NS}_\kappa^{\aleph_0}$ collapses λ^+ to λ.

Our basic assumption will be that there is no inner model of $\exists \alpha\ O(\alpha) = \alpha^{++}$.

Suppose that some regular cardinal κ in $\mathcal{K}(\vec{\mathcal{F}})$ changing its cofinality still remains of cardinality above $2^{\aleph_0}$. We will consider elementary submodels $N \prec H_\mu$ for some $\mu > \kappa$ such that ${}^\omega N \subseteq N$, $|N| < |\kappa|$, $\kappa \in N$ and $\bigcup(N \cap \kappa) = \kappa$. By Mitchell [Mi2–4] there is a function $h^N \in \mathcal{K}(\vec{\mathcal{F}})$, a sequence of indiscernibles $\mathbb{C}^N$ and $\rho < |N|^+$ such that $N \cap H_\kappa \cap \mathcal{K}(\vec{\mathcal{F}})$ is contained in $h^{N\prime\prime}(\rho; \mathbb{C}^N)$. Let us refer to Mitchell's [Mi4] for basic definitions and facts on such models.

By a submodel we will always mean some N as above. Denote by $\bar{\kappa}^N$ the least ordinal in $h^{N\prime\prime}(\kappa)$. We will often drop the upper index N when it will be clear to which N everything is related.

LEMMA 2.1: *Suppose that κ is a regular cardinal in $\mathcal{K}(\vec{\mathcal{F}})$ which became singular in V. Assume that for some λ, $|\kappa| > 2^\lambda$ and $\mathrm{cf}\,\kappa \le \lambda^+$. Suppose that C is a club of κ in a model $V_1 \models ZFC + \kappa$ is regular, $\mathcal{K}(\vec{\mathcal{F}}) \subseteq V_1 \subseteq V$, such that all the points of C of cofinality ν are regular in $\mathcal{K}(\vec{\mathcal{F}})$, for some $\nu \le \lambda^+$. Then there exists a cofinal in κ sequence $\langle \tau_i \mid i < \mathrm{cf}\ \kappa\rangle$ so that for all $i < \mathrm{cf}\ \kappa$*

(i) $\tau_i \in C$,

(ii) $\bigcup(C \cap \tau_i) = \tau_i$,

(iii) $\mathrm{cf}\ \tau_i \ge \lambda^{++}$,

(iv) *for every submodel N with $C, \langle \tau_i \mid i < \omega\rangle \in N$ there is $i_0 < \omega$ such that all τ_i's $(i \ge i_0)$ are limit indiscernibles for $\bar{\kappa}^N$ (i.e., there are unboundedly many indiscernibles for $\bar{\kappa}^N$ in $\bigcup(N \cap \tau_i)$).*

Remark: As the referee of the paper pointed out, the assumption on C can be weakened by removing V_1 and requiring that $otp(C \cap D) = \kappa$ for every club $D \in K(\vec{\mathcal{F}})$.

Proof: Let N be a submodel so that $C \in N$ and ${}^\lambda N \subseteq N$.

For an ordinal $\beta < \kappa$ set $\beta^{(0)} = \beta$, $\beta^{(n+1)} = \bigcup\left(h''(\beta^{(n)}+1) \cap \kappa\right)$ and $\beta^{(\omega)} = \bigcup_{n<\omega} \beta^{(n)}$. Define a club C_h in $\mathcal{K}(\vec{\mathcal{F}})$ as follows:

$$C_h = \{\alpha < \kappa \mid \text{for every } \beta < \alpha, \quad \beta^{(\omega)} < \alpha\}\ .$$

Pick $\delta < \kappa$ big enough to that there are no indiscernibles for ordinals $> \bar{\kappa}$ inside the interval $[\delta, \kappa)$. It is possible since indiscernibles for $\bar{\kappa}$ are unbounded in κ; see [Mi3, Lemma 6.3].

CLAIM 2.1.1: *Let $\tau^* < \kappa$ be an indiscernible for $\overline{\kappa}$ and τ^{**} be the least indiscernible for $\overline{\kappa}$ above it. Then $(\bigcup(N \cap \tau^{**}) \cap (C_h - (\tau^* + 1)) = \emptyset$ and $(\tau^*+1)^{(\omega)} \ge \bigcup(N \cap \tau^{**})$.*

Proof: It is enough to show that $(\tau^*+1)^{(\omega)} \ge \bigcup(N \cap \tau^{**})$. Let us prove by induction that for every $\xi \in N$, $\tau^* \le \xi < \tau^{**}$ there is n such that $\bigcup\left(h''(\xi+1) \cap \tau^{**}\right) \le (\tau^*+1)^{(n)}$.

Suppose that this holds for every $\xi' < \xi$. If ξ is not indiscernible, then for some $\xi' < \xi$, $\xi \in h''(\xi'+1) \cap \tau^{**}$. So $\xi \le (\tau^*+1)^n$ for some $n < \omega$. Hence $\bigcup\left(h''(\xi+1)\cap\tau^{**}\right) \le (\tau^*+1)^{n+1}$. If ξ is an indiscernible, then it is an indiscernible for some $\alpha(\xi) \in h''(\xi'+1)$ where $\xi' < \xi$. But then $\xi < \alpha(\xi) < \bigcup(h''(\xi'+1) \cap \tau^{**}) \leq (\tau^*+1)^n$ for some n. ∎

Let C' be the set of limit points of C. Consider $C' \cap C_h$. It is a club in V_1. Let τ^* be an indiscernible for $\bar{\kappa}$ so that $\kappa > \tau^* > \max(\delta, (2^\lambda)^+)$. Let τ^{**} be the least indiscernible for $\bar{\kappa}$ above τ^*.

CLAIM 2.1.2: $C_h \cap C' \cap (\tau^*, \tau^{**}] = \emptyset$.

Proof: Suppose otherwise. First note that C' is unbounded in τ^{**}. Since otherwise $\bigcup(C' \cap \tau^{**}) \in N$ and by Claim 2.1.1 $(\bigcup(N \cap \tau^{**})) \cap (C_h - (\tau^*+1)) = \emptyset$.

Let $\mu = \bigcup(C \cap \tau^{**} \cap N) = \bigcup(\tau^{**} \cap N)$. By the proof of Claim 2.1.1, μ is a singular cardinal in $\mathcal{K}(\vec{\mathcal{F}})$. cf $\mu \geq \lambda^+ > \aleph_0$, since ${}^\lambda N \subseteq N$. Recall that all points of C of some fixed cofinality $\nu \leq \lambda^+$ are regular in $\mathcal{K}(\mathcal{F})$. Then cf $\mu > \nu$, since $\mu \in C$ and it is singular in $\mathcal{K}(\vec{\mathcal{F}})$. Let $C_1 \in \mathcal{K}(\vec{\mathcal{F}})$ be a club in μ consisting of singular in $\mathcal{K}(\vec{\mathcal{F}})$ ordinals. Then $C \cap C_1$ contains an element of cofinality ν. But every such element of C is regular in $\mathcal{K}(\vec{\mathcal{F}})$. Contradiction. ∎

Hence above $\max(\delta, (2^\lambda)^+)$ every element of $N \cap C_h \cap C'$ is a limit indiscernible for $\bar{\kappa}$.

Let us turn now to the construction of the sequence $\langle \tau_i \mid i < \text{cf } \kappa \rangle$. We shall find one τ satisfying the conditions (i), (ii) and (iii) for N. It will be clear that it is possible to find such τ above any $\rho < \kappa$.

Let τ be the least indiscernible for $\bar{\kappa}$ so that $\text{otp}(C' \cap C_h \cap \tau) > (2^\lambda)^+$. Then by Claim 2.1.2, τ is a limit indiscernible for $\bar{\kappa}$ and $\tau \in C' \cap C_h$. If cf $\tau \leq \lambda^+$, then N contains a cofinal in τ sequence. So there is a sequence $\{\tau_i \mid i < \text{cf } \tau\} \subseteq N$ of indiscernibles for $\bar{\kappa}$ unbounded in τ. But then for some $i_0 <$ cf τ, $\text{otp}(C' \cap C_h \cap \tau_{i_0}) > (2^\lambda)^+$, which contradicts the choice of τ.

Hence cf $\tau \geq \lambda^{++}$.

Let show now that the sequence $\langle \tau_i \mid i < \text{cf } \kappa \rangle$ which was constructed for N is good for any other submodel N' with $\langle \tau_i \mid i < \text{cf } \kappa \rangle \in N'$.

Pick a submodel N^* which contains $N, N', \{h^N\}, \{h^{N'}\}$. By Mitchell [Mi4], for some $i_0 <$ cf κ all τ_i's with $i \geq i_0$ are indiscernibles for $\bar{\kappa}^{N^*}$ in N^*. It implies that for some $i_1 <$ cf κ all τ_i's with $i \geq i_1$ are indiscernibles for $\bar{\kappa}^{N'}$ in N'. ∎

LEMMA 2.2: *Suppose that κ is a strong limit cardinal singular in V but regular in $\mathcal{K}(\vec{\mathcal{F}})$. Suppose that C is a club of κ in a model V_1, $\mathcal{K}(\vec{\mathcal{F}}) \subseteq V_1 \subseteq V$, $V_1 \models ZFC + \kappa$ is regular such that all the points of C of some fixed cofinality ν*

are regular in $\mathcal{K}(\vec{\mathcal{F}})$. Then there exists a cofinal in κ sequence $\langle \tau_i \mid i < \text{cf } \kappa\rangle$ so that for all $i < \text{cf } \kappa$

(i) $\tau_i \in C$,

(ii) $\bigcup(C \cap \tau_i) = \tau_i$,

(iii) *the set $\{\text{cf } \tau_i \mid i < \text{cf } \kappa\}$ is unbounded in κ,*

(iv) *for every submodel N with C, $\langle \tau_i \mid i < \text{cf } \kappa\rangle \in N$ there is $i_0 < \text{cf } \kappa$ such that all τ_i's $(i \geq i_0)$ are limit indiscernibles for $\bar{\kappa}^N$.*

Proof: Let $\langle \lambda_i \mid i < \text{cf } \kappa\rangle$ be a cofinal sequence in κ. For every $i < \text{cf } \kappa$ use Lemma 2.1 to pick a sequence $\langle \tau^i_j \mid j < \text{cf } \kappa\rangle$ satisfying (i)–(iv) for $\lambda = \lambda_i$. Let N be a submodel containing all the sequences $\langle \tau^i_j \mid j < \text{cf } \kappa\rangle$, $i < \text{cf } \kappa$. For every i let $j(i) < \text{cf } \kappa$ be such that $\langle \tau^i_j \mid j \geq j(i)\rangle$ is a sequence of indiscernibles for $\bar{\kappa}^N$ and $\tau^i_{j(i)} > \lambda_i$. Set $\tau_i = \tau^i_{j(i)}$ for every i. Clearly $\langle \tau_i \mid i < \text{cf } \kappa\rangle$ satisfies the conditions (i)–(iii). But (iv) is also satisfied since $\langle \tau_i \mid i < \text{cf } \kappa\rangle$ is a sequence of indiscernibles for $\bar{\kappa}$ in N which is unbounded in κ. ∎

LEMMA 2.3: *Let κ, C be as in Lemma 2.2. Suppose that N is a submodel containing C, $\langle \tau_i \mid i < \text{cf } \kappa\rangle$ and so that ${}^\lambda N \subseteq N$ for some regular λ. Then $C \cap N$ contains unboundedly many indiscernibles for $\bar{\kappa}^N$ of cofinality λ.*

Proof: It follows from Lemmas 2.1 and 2.2, since $\text{cf}\left(\bigcup(N \cap \tau_i)\right) \geq \lambda^+$ and the indiscernibles for $\bar{\kappa}^N$ form a λ-club by (iv) and [Mi4]. ∎

LEMMA 2.4: *Let κ, C be as in Lemmas 2.1 or 2.2. Let λ be a regular cardinal such that $|\kappa| > 2^\lambda$. Then there exists a submodel N so that $C \cap N$ contains unboundedly many indiscernibles for $\bar{\kappa}^N$ of every cofinality $\leq \lambda^+$.*

Proof: By Lemma 2.3, it is enough to find indiscernibles of cofinality λ^+. Pick the sequence $\langle \tau_i \mid i < \text{cf } \kappa\rangle$ as in Lemma 2.2. Let N^* be a submodel containing C, $\langle \tau_i \mid i < \text{cf } \kappa\rangle$ and such that ${}^\lambda N^* \subseteq N^*$. Find a submodel N such that $N \supseteq N^* \cup \{h^{N^*}\}$ and ${}^\lambda N \subseteq N$. Without loss of generality assume that $\langle \tau_i \mid i < \text{cf } \kappa\rangle$ is a sequence of indiscernibles for $\bar{\kappa}^N$ in N and for $\bar{\kappa}^{N^*}$ in N^*. Set $\mu_i = \bigcup(N^* \cap \tau_i)$ for $i < \text{cf } \kappa$. If $\text{cf } \mu_i > \lambda^+$, then $C \cap \mu_i$ contains elements of cofinality λ^+. Assume that for every $i < \text{cf } \kappa$ $\text{cf } \mu_i = \lambda^+$. It is enough to show that a final segment of $\langle \mu_i \mid i < \text{cf } \kappa\rangle$ consists of indiscernibles for $\bar{\kappa}^N$ in N. Suppose otherwise. Then for an unbounded set $I \subseteq \text{cf } \kappa$ all μ_i $(i \in I)$ are not indiscernibles for $\bar{\kappa}^N$. By removing an initial segment of I, we can assume then that μ_i's are indiscernibles for ordinals below $\bar{\kappa}^N$. Pick for every μ_i some $\nu_i < \mu_i$

such that $\bigcup(h^{N''}(\nu_i)\cap\kappa) > \mu_i$. For every $i \in I$ there exists an indiscernible ξ_i for $\bar{\kappa}^{N^*}$, $\nu_i < \xi_i < \mu_i$, by (iv) of Lemmas 2.1 and 2.2. Now a final segment of $\langle \xi_i \mid i \in I\rangle$ are still indiscernibles for $\bar{\kappa}^N$ in N, by [Mi4]. But it is impossible since $\bigcup(h^{N''}(\nu_i)\cap\kappa) > \mu_i > \xi_i$. Contradiction.

The following is a standard fact about generic ultrapowers and it will be used frequently below.

LEMMA 2.4.1: *Suppose F is a precipitous filter over κ. Let $\mathcal{U}$ be a generic ultrapower extending F and i_u: $V \to M = ult(V,\mathcal{U})$ the corresponding generic ultrapower.*

(1) *Then κ is the critical point of i_u.*

(2) *If for some δ, $\{\alpha < \kappa \mid \operatorname{cf}\alpha = \delta\} \in \mathcal{U}$, then, in M, $\operatorname{cf}\kappa = \delta$.*

THEOREM 2.5*: *Suppose GCH. If NS^μ_κ is precipitous for $\kappa > \aleph_2$ and $\mu^+ < \kappa$, then there exists a weak repeat point in $\mathcal{K}(\vec{\mathcal{F}})$.*

Proof: By Mitchell [Mi2], we can assume that κ is an inaccessible or the successor of a regular cardinal. Consider the set $S = \{\alpha < O^{\vec{\mathcal{F}}}(\kappa) \mid$ some $A \in (\mathrm{NS}^\mu_\kappa)^+$ forces that the measure $\mathcal{F}(\kappa,\alpha)$ is used to move κ in the generic ultrapower$\}$, i.e. it is used first in the ultrapower of $\mathcal{K}(\vec{\mathcal{F}})$.

Let $\alpha = \min S$. Suppose that α is not a weak repeat point. Then pick a set $A \in \mathcal{F}(\kappa,\alpha)$ consisting of regular cardinals in $\mathcal{K}(\vec{\mathcal{F}})$ such that $A \notin \mathcal{F}(\kappa,\alpha')$ for $\alpha' < \alpha$. Consider the set $B = \{\delta < \kappa \mid \delta$ is a regular in $\mathcal{K}(\vec{\mathcal{F}})$ and there is $\delta' < O^{\vec{\mathcal{F}}}(\delta)$ so that $A\cap\delta \in \mathcal{F}(\kappa,\delta')\}$.

Then $A\cup B \in \bigcap_{\beta\geq\alpha}\mathcal{F}(\kappa,\beta)$ and $A\cup B \notin \bigcup_{\beta<\alpha}\mathcal{F}(\kappa,\beta)$. Since $\alpha = \min S$, $A\cup B$ contains a μ-club. Let C be a club so that its points of cofinality μ are in $A\cup B$. Then every $\delta \in C$ of cofinality μ will be regular in $\mathcal{K}(\mathcal{F})$.

Let M be a generic ultrapower defined by a generic embedding using the measure $\mathcal{F}(\kappa,\alpha)$. Then $C, A, B \in M$. By Mitchell [Mi2], $\mathcal{F}^M \mid (\kappa+1) = \mathcal{F} \mid (\kappa,\alpha)$, and $\mathcal{P}(\kappa)\cap\mathcal{K}(\vec{\mathcal{F}}) = \mathcal{P}(\kappa)\cap\mathcal{K}(\mathcal{F}^M)$ where $\mathcal{F}^M$ is the maximal sequence for the core model inside M.

Now apply the previous lemmas inside M where $V_1 = \mathcal{K}(\mathcal{F}^M)(C)$. Pick N to be as in Lemma 2.4. Assume that $A\cup B \in N$. Then, by Lemma 2.4, there exists an indiscernible in C for $\bar{\kappa}^N$ of cofinality μ above the support of $A\cup B$. But it implies (see [Mi4]) that $A\cup B \in \mathcal{F}(\kappa,\beta)$ for some $\beta < O^{\mathcal{F}^M}(\kappa) = \alpha$, which is impossible. Contradiction. ∎

* A version of this theorem was proved jointly with M. Magidor.

Suppose now that $\mathrm{NS}_\kappa^{\aleph_0}$ is precipitous for $\kappa > (2^{\aleph_0})^+$. Let us show that there exists more than just a weak repeat point in $\mathcal{K}(\vec{\mathcal{F}})$. Our aim will be to prove the following:

THEOREM 2.5.1: *Suppose that $\mathrm{NS}_\kappa^{\aleph_0}$ is precipitous for $\kappa > (2^{\aleph_0})^+$. Then*

(a) *if $\kappa = \lambda^{++}$ for λ of cofinality $> \omega$ and $\lambda^\omega = \lambda$, then there exists an (ω, λ^+)-repeat point,*

(b) *if $\kappa = \lambda^+$ for a weakly inaccessible λ so that $\mu^\omega < \lambda$ for every $\mu < \lambda$, then there exists an (ω, λ)-repeat point,*

(c) *(Mitchell [Mi2]) if κ is a successor of a singular cardinal, then $\exists\kappa\, O(\kappa) = \kappa^{**}$ in an inner model,*

(d) *if κ is a weakly inaccessible and $\mu^\omega < \kappa$ for every $\mu < \kappa$ then there exists an $(\omega, < \kappa)$-repeat point,*

(e) *if $\kappa = \lambda^{++}$ for λ of cofinality ω so that $\mu^\omega < \lambda$ for every $\mu < \lambda$, then there exists an $(\omega, < \lambda)$-repeat point.*

By Mitchell's covering lemma, for any $N \prec H_{\kappa^+}$ such that ${}^\omega N \subseteq N$, $|N| < |\kappa|$ and $N \cap \kappa$ is cofinal in κ, $N \cap H_\kappa \cap \mathcal{K}(\vec{\mathcal{F}})$ is contained in $h^{N\prime\prime}(\rho^N; \mathbb{C}^N)$, where $\rho^N < |N|^+$, h^N is the Skolem function and $\mathbb{C}^N$ the sequence of indiscernibles. The sequence $\mathbb{C}^N$ consists of critical points of the iteration of the least missing in N mouse (more precisely, in the transitive collapse of N). It turns out [Mi4] that the measures of the mouse below κ are the right one, i.e. they are on the sequence $\vec{\mathcal{F}}$. It is not true in general over κ itself.

But if it is possible to pick $N \prec H_{\lambda^+}$ for some $\lambda > \kappa$ which also is changing its cofinality such that $\kappa \in N$, $N \cap \kappa$ is cofinal in κ, $|N| < \kappa$ and $\kappa > \rho^N$, then $N \cap H_\lambda \cap \mathcal{K}(\vec{\mathcal{F}}) \subseteq h^{N\prime\prime}(\rho^N; \mathbb{C}^N)$ and the measures over κ given by the mouse are the right ones. Let us call an ordinal δ (like κ above) satisfying this a good ordinal and a submodel witnessing it a good submodel for δ. By the proof of Lemma 2.1, all interesting δ's in C of the lemma are good, i.e., for $N \prec H_{\kappa^+}$, $\delta > \rho^N$. Let us use this in order to find a generic ultrapower in which κ is good.

Let ν be such that $\kappa = \nu^{++}$ and $\operatorname{cf}\nu > \aleph_0$ or $\nu = 2^{\aleph_0}$ otherwise. Set $Y = \{\delta < \kappa$: $\operatorname{cf}\delta = \omega$ and every submodel of cardinality $\leq \nu$ is contained in a good for δ submodel$\}$.

CLAIM: *Y is stationary.*

Proof: Suppose otherwise. Let E be a club in κ disjoint to Y. For every δ in E, $\operatorname{cf}\delta = \omega$, let N_δ be a model which is not contained in any good for δ submodel.

Now use a club C and a submodel N as in the proof of Lemma 2.1 with E and the function taking δ to N_δ in N. The contradiction follows now easily. ∎

Let us call α a relevant ordinal and $\mathcal{F}(\kappa, \alpha)$ a relevant measure, if some condition stronger than Y forces that the measure $\mathcal{F}(\kappa, \alpha)$ is used first in the generic embedding to move κ. Clearly, $Y \Vdash$"$\check{\kappa}$ is good in the generic ultrapower and every submodel of cardinality $\leq \nu$ is contained in a good submodel of cardinality ν there" where $\nu^{++} = \kappa$ or $\nu = 2^{\aleph_0}$.

Let $B \in \bigcap\{\mathcal{F}(\kappa, \gamma) \mid \gamma \text{ is a relevant ordinal}\}$. Then $(\kappa - Y) \cup (B \cap Y)$ contains a club, since otherwise $Y - B = X$ is stationary. Pick some generic ultrafilter G with $X \in G$. Let $j_G\colon V \to M \simeq V^\kappa/G$ be the generic elementary embedding. Find α such that $\mathcal{F}(\kappa, \alpha)$ was used first to move κ. But then α is a relevant ordinal. So $B \in \mathcal{F}(\kappa, \alpha)$. But then $\kappa \in j_G(B)$ and $\kappa \in j_G(X)$. Hence $B \cap X \neq \emptyset$. Contradiction.

Further, let us always assume for $B \in \bigcap\{\mathcal{F}(\kappa, \gamma) \mid \gamma \text{ is a relevant ordinal}\}$ that $B \subseteq Y$. Also by a club C contained in B we mean $C \subseteq (\kappa - Y) \cup (B \cap Y)$.

For a model N as in the Mitchell covering lemma we are going to use the function $\beta^N(-)$ and the coherence function $\mathbb{C}^N(-,-,-)$ for $\vec{\mathcal{F}}$ introduced in [Mi2-4]. For an indiscernible $\tau \in N$, $\beta^N(\tau)$ denotes the measure over $\overline{\kappa}^N$ for which τ is an indiscernible (more precisely, the index of this measure). Recall that $\overline{\kappa}^N$ is the least ordinal in h^N"(κ). The coherence function $\mathbb{C}^N(\overline{\kappa}^N, \beta^N(\tau), \beta^N(\kappa))(\kappa)$ gives the ordinal ξ s.t. τ is an indiscernible for $\mathcal{F}(\kappa, \xi)$.

Let α be a relevant ordinal. Force with $\mathrm{NS}_\kappa^{\aleph_0}$. Let $i_G\colon V \longrightarrow M = ult(V, G) \subseteq V[G]$ be a generic embedding such that $\mathcal{F}(\kappa, \alpha)$ is used first to move κ in $i_G \restriction \mathcal{K}(\vec{\mathcal{F}})$. Then, in M, κ changes its cofinality to $\aleph_0$ and the Mitchell covering lemma can be applied there. Note that $\mathcal{P}(\kappa) \cap V \subseteq \mathcal{P}(\kappa) \cap M$. Working in M we considered an ordinal $\overline{\alpha}$ which is the supremum of all $\beta < \alpha = 0^{i_G(\vec{\mathcal{F}})}(\kappa)$ satisfying the following:

For every $B \in \bigcap\{\mathcal{F}(\kappa, \gamma) \mid \gamma \text{ relevant}\}$ and every submodel N' of H_{λ^+} there are a club in V, $C \subseteq B$, and a good for κ submodel $N \supset N'$ of H_{λ^+} so that there exists a set $s \subseteq N \cap (\alpha - \beta)$, $|s| > \aleph_0$ s.t. for every $\xi \in s$, N has unboundedly many in κ indiscernibles $\tau \in C$ for $\bar{\kappa}^N$ satisfying $\mathbb{C}^N(\bar{\kappa}^N, \beta^N(\tau), \beta^N(\kappa))(\kappa) \geq \xi$ and $\mathrm{cf}^M \tau = \aleph_0$.

Note that by Lemma 2.3, $\underline{\alpha} > 0$. Let $\underset{\sim}{\alpha}$ be a name of such $\underline{\alpha}$.

For a relevant ordinal α set $\hat{\alpha} = \min\{\beta \mid \text{some condition forces "}\check{\beta} = \underset{\sim}{\alpha}\text{"}\}$. Let $\alpha^* = \min\{\hat{\alpha} \mid \alpha \text{ is a relevant ordinal}\}$. Denote by α^0 the least relevant α so that

$\hat{\alpha} = \alpha^*$.

Let $\alpha^{\min}$ be the least relevant ordinal. Clearly, $\alpha^* \leq \alpha^{\min} \leq \alpha^0$.

By the choice of α^* and [Mi4, Lemma 1.6], there exist a set $B \in \bigcap\{\mathcal{F}(\kappa,\gamma) \mid \gamma$ is a relevant ordinal$\}$, a submodel N' of $H^M_{\lambda^+}$ and $\xi < \kappa$ so that for every club $C \in V$, $C \subseteq B$ and every good for κ submodel $N \supseteq N'$ of $H^M_{\lambda^+}$ there are only countably many indiscernibles τ, $\xi < \tau \in C$ for $\bar{\kappa}^N$ satisfying $\mathbb{C}^N(\bar{\kappa}^N, \beta^N(\tau), \beta^N(\kappa))(\kappa) \geq \alpha^*$ and cf $\tau = \omega$. Let us always assume below that every set picked in $\bigcap\{\mathcal{F}(\kappa,\gamma) \mid \gamma$ is a relevant ordinal$\}$ is a subset of this particular set B, all the indiscernibles are above ξ and are outside of this countably many.

Further, by a good submodel we shall mean a good submodel for κ.

Let us assume that there is no up-repeat point.

LEMMA 2.6: *If the ordinal $\alpha^{\min} - \alpha^*$ is uniformly representable (i.e., it can be represented by the same function in all ultrapowers of $\mathcal{K}(\vec{\mathcal{F}})$ by $\mathcal{F}(\kappa,\beta)$ for $\beta < O^{\vec{\mathcal{F}}}(\kappa)$), then α^* is $(\alpha^{\min} - \alpha^*) + 1$-repeat point and every $A \in \bigcap\{\mathcal{F}(\kappa,\beta) \mid \alpha^* \leq \beta \leq \alpha^{\min}\}$ reflects to unboundedly many in α^* places.*

Proof: Suppose that $g\colon \kappa \to \kappa$ uniformly represents $\alpha^{\min} - \alpha^*$. Let $A \in \bigcap\{\mathcal{F}(\kappa,\beta) \mid \alpha^* \leq \beta \leq \alpha^{\min}\}$ and $\alpha' < \alpha^*$. Pick a set $A' \in \mathcal{F}(\kappa,\alpha')$ which does not belong to any measure above α'. Set $B = \{\delta < \kappa \mid$ there is $\delta^* < O^{\vec{\mathcal{F}}}(\delta)$ so that $\delta^* + g(\delta) < O^{\vec{\mathcal{F}}}(\delta)$, $A \cap \delta \in \bigcap\{\mathcal{F}(\delta,\beta) \mid \delta^* \leq \beta \leq \delta^* + g(\delta)\}$ and there is $\delta' < \delta$ so that $A' \cap \delta \in \mathcal{F}(\delta,\delta')$ and $A' \cap \delta$ does not belong to any measure on δ above $\delta'\}$.

Let A_1 have the same definition as B but only $\delta^* + g(\delta) = O^{\vec{\mathcal{F}}}(\delta)$ and $\delta \in A$. Then B belongs to every measure $\mathcal{F}(\kappa,\beta)$ with $\beta > \alpha^{\min}$ and $A_1 \in \mathcal{F}(\kappa,\alpha^{\min})$. So $A_1 \cup B \in \bigcap\{\mathcal{F}(\kappa,\alpha) \mid \alpha$ is a relevant measure$\}$. Let $G \subseteq (\mathrm{NS}^{\aleph_0}_\kappa)^+$ be a generic ultrafilter such that $j_G\colon V \to M \simeq V^\kappa/G$ uses $\mathcal{F}(\kappa,\alpha^0)$ to move κ and $\underset{\sim}{\alpha}^0[G] = \alpha^*$. By the choice of α^*, there are a club $C \subseteq \kappa$, $C \in V$, points of cofinality ω of C are in $A_1 \cup B$ and a good submodel N of $H^M_{\lambda^+}$ so that unboundedly many indiscernibles $\tau \in C$ for $\bar{\kappa}^N$ of cofinality ω come from the measures above α' and below α^*. Then $A_1 \cup B \in \mathcal{F}(\kappa,\beta)$ for some $\alpha' < \beta < \alpha^*$. So $A_1 \in \mathcal{F}(\kappa,\beta)$ or $B \in \mathcal{F}(\kappa,\beta)$. But in any case the definition of A_1, B implies that $A \in \bigcap\{\mathcal{F}(\kappa,\gamma) \mid \beta' \leq \gamma \leq \beta' + (\alpha^{\min} - \alpha^*)\}$ for some β', $\alpha' < \beta' < \beta$. ∎

The following lemma has the similar proof.

LEMMA 2.7: *If an ordinal γ is uniformly representable and $\alpha^* + \gamma \leq \alpha^{\min}$, then α^* is a $(\gamma+1)$-repeat point and every $A \in \bigcap\{\mathcal{F}(\kappa,\beta) \mid \alpha^* \leq \beta \leq \alpha^* + \gamma\}$ reflects* *to unboundedly many in α^* places.*

We are interested in showing the existence of γ-repeat points for $\gamma \leq \kappa^+ + \kappa^+$. Clearly, there is no problem to find uniformly representing functions for such ordinals.

Suppose now that there is no $(\omega, \kappa^+ + \kappa^+)$-repeat point.

If $\omega < \mathrm{cf}\ \alpha^* < \kappa^+$, then by 1.15 there is an $(\omega, \kappa^+ + \kappa^+)$-repeat point.

If $\mathrm{cf}\ \alpha^* = \kappa^+$ and there is an increasing sequence $\langle \beta_n \mid n < \omega \rangle$ of ordinals below α^* so that every $A \in \mathcal{F}(\kappa, \alpha^*)$ belongs to some measure in every interval (β_n, β_{n+1}), then by Lemma 1.16 there exists an $(\omega, \kappa^+ + \kappa^+)$-repeat point. Hence there is a maximal set $\{\beta_0, \ldots, \beta_{n-1}\}$ such that every $A \in \mathcal{F}(\kappa, \alpha^*)$ belongs to some measure in every interval (β_i, β_{i+1}) $(i < n)$ and for every $\beta \notin \{\beta_0, \ldots, \beta_{n-1}\}$ there exists some $A_\beta \in \mathcal{F}(\kappa, \alpha^*)$ such that $A_\beta \notin \bigcup \{\mathcal{F}(\kappa, \gamma) \mid \beta > \gamma > \max\{\beta_i \mid \beta_i < \beta\}$.

Let us choose a set $A^* \in \mathcal{F}(\kappa, \alpha^*)$ which does not belong to any measure above α^*.

Let B be the set consisting of all δ's below κ so that

(1) $O^{\vec{\mathcal{F}}}(\delta) > 0$,

(2) $\mathrm{cf}\ \delta = \omega$,

(3) there exists the maximal $\delta^* < O^{\vec{\mathcal{F}}}(\delta)$ such that

  (a) $A^* \cap \delta \in \mathcal{F}(\delta, \delta^*)$,

  (b) $\mathrm{cf}\ \delta^* = \omega$, if $\mathrm{cf}\ \alpha^* = \omega$ and $\mathrm{cf}\ \delta^* = \delta^+$, if $\mathrm{cf}\ \alpha^* = \kappa^+$,

  (c) for every $\beta < \delta^*$ there exists a good submodel N, for δ, and there are unboundedly many in δ indiscernibles τ for $\bar{\delta}^N$ such that

$$\mathbb{C}^N(\bar{\delta}^N, \beta^N(\tau), \beta^N(\delta))(\delta) \geq \beta$$

  (where $\bar{\delta}^N$ is the least ordinal $\geq \delta$ in $h^{N''}(\delta)$).

By the definition of α^*, $A^* \cup B \in \bigcap\{\mathcal{F}(\kappa, \alpha) \mid \alpha \text{ is a relevant ordinal}\}$. Then there exists a club $C \in V$ whose points of cofinality ω are in $A^* \cup B$. Notice that if $\alpha^* < \alpha^{\min}$, then B alone contains an ω-club.

Let $j: V \to M$ be a generic elementary embedding witnessing "$\check{\alpha}^* = \underset{\sim}{\alpha}^0$". Recall that $\mathrm{cf}\, \kappa = \aleph_0$ in M.

Pick in M a good submodel $N \prec H_{\lambda^+}$ so that

(a) $|N| = 2^{\aleph_0}$, $N \supseteq 2^{\aleph_0}$ if κ is an inaccessible or $\kappa = \nu^{++}$ for singular ν and $|N| = \nu$, $N \supseteq \nu$ if $\kappa = \nu^{++}$ for a regular ν,

(b) ${}^{\omega}N \subseteq N$,

(c) α^* , $\kappa \in N$,

(d) A^* , B , $C \in N$,

(e) there exists $\delta < \kappa$ so that for every indiscernible $\tau \in C - \delta$ for $\bar{\kappa}^N$ with cf $\tau = \aleph_0$, for every ordinal $\beta \in \beta^N(\tau) \cap N$, either

(i) $\tau \in B$ and $\mathbf{C}^N(\bar{\kappa}^N, \beta, \beta^N(\tau))(\tau) \geq \tau^*$

or

(ii) for every countable $x \subseteq \beta^N(\tau) - \beta$ there exist unboundedly many in τ indiscernibles τ_1 for $\bar{\kappa}^N$ such that $\beta < \beta^N(\tau_1) < \beta^N(\tau)$ and

$$\mathbf{C}^N(\bar{\kappa}^N, \beta, \beta^N(\tau))(\tau) < \mathbf{C}^N(\bar{\kappa}^N, \beta^N(\tau_1), \beta^N(\tau))(\tau) \notin x,$$

where τ^* is the largest ordinal $< O^{\vec{\mathcal{F}}}(\tau)$ s.t. $A^* \cap \tau \in \mathcal{F}(\tau, \tau^*)$.

The only nontrivial condition on N is (e). Since $\tau \in C$ some good for τ $N_\tau \prec H_{\lambda^+}$ satisfies (e) for unboundedly many in τ indiscernibles τ_2. N can be picked as a good submodel of H_{μ^+} satisfying (a)–(d) for μ big enough in order to catch the function $\tau \to N_\tau$. Such N will satisfy (e) since for every τ as in (e) unboundedly many in τ τ_2's will be indiscernibles for the same measures in N and in N_τ, by [Mi4, Lemma 1.5].

Let $\langle \tau_n \mid n < \omega \rangle$ be the sequence of indiscernibles given by Lemma 2.1 and C_h be as in Lemma 2.1. We are dropping the index N for a while. Suppose that $\tau_0 > \delta$ for δ as in (e). For $n < \omega$, let d_n be the ω-club in $\bigcup(N \cap \tau_n)$ consisting of indiscernibles of cofinality ω in $C \cap C_h \cap \tau_n$ for $\bar{\kappa}$. Since cf $\tau_n > \omega$, by removing an initial segment, we can assume that for every $\tau < \tau'$ in d_n and an indiscernible for $\bar{\kappa}$ τ'' s.t. $\tau < \tau'' < \tau'$, $\beta(\tau) < \beta(\tau')$ and $\beta(\tau'') < \beta(\tau')$. It follows by [Mi4, Lemma 1.6].

For an indiscernible $\mu \in C$ for $\bar{\kappa}$ such that cf $\mu = \aleph_0$, let $\mu^* = O^{\vec{\mathcal{F}}}(\mu)$ if $\mu \notin B$, otherwise let μ^* be the largest ordinal $< O^{\vec{\mathcal{F}}}(\mu)$ such that $A^* \cap \mu \in \mathcal{F}(\mu, \mu^*)$. Denote by $\beta^*(\mu)$ the corresponding to μ^* over $\bar{\kappa}$, i.e. ordinal or more precisely the index of the measure β such that $\mu^* = \mathbb{C}(\bar{\kappa}, \beta, \beta(\mu))(\mu)$.

LEMMA 2.8: *Suppose that* cf $\alpha^* = \kappa^+$. *For every* $n < \omega$, *for every increasing sequence* $\langle \mu_{mn} \mid m < \omega \rangle$ *of elements of* d_n

$$\beta^*\left(\bigcup_{m<\omega} \mu_{mn}\right) < \bigcup_{m<\omega} \beta(\mu_{mn}) \leq \beta\left(\bigcup_{m<\omega} \mu_{mn}\right).$$

Proof: The right inequality holds by [Mi4, Lemma 1.6]. Let us prove the left inequality. Suppose otherwise. Then

$$\beta^*\left(\bigcup_{m<\omega} \mu_{mn}\right) \geq \bigcup_{m<\omega} \beta(\mu_{mn}) .$$

Denote $\bigcup \mu_{mn}$ by μ. Then $\mathrm{cf}\,\mu = \omega$. By the Weak Covering Lemma [Mi1], $\mathrm{cf}\,\mu^+ > \omega$. Then $\mathrm{cf}\,\alpha^* = \kappa^+$ implies $\mathrm{cf}\,\mu^* > \omega$. Let us turn now to μ_{mn}'s. Since they form an increasing sequence in d_n, the sequence $\langle \beta(\mu_{mn}) | m < \omega \rangle$ is also increasing. We replace every $\beta(\mu_{mn})$ by the corresponding ordinal over μ, i.e. $\mathbb{C}(\overline{\kappa}, \beta(\mu_{mn}), \beta(\mu))(\mu)$. Denote $\mathbb{C}(\overline{\kappa}, \beta(\mu_{mn}), \beta(\mu))(\mu)$ by γ_m, and let $\gamma = \bigcup_{m<\omega} \gamma_m$. Then $\langle \gamma_m | m < \omega \rangle$ is an increasing sequence, and so $\mathrm{cf}\,\gamma = \omega$. Since we assumed $\beta^*(\mu) \geq \bigcup_{m<\omega} \beta(\mu_{mn})$, by transferring this to μ we obtain $\mu^* \geq \gamma$. Recall that $\beta^*(\mu)$ is defined to be the ordinal β such that $\mu^* = \mathbb{C}(\overline{\kappa}, \beta, \beta(\mu))(\mu)$. But since $\mathrm{cf}\,\mu^* \neq \mathrm{cf}\,\gamma = \omega$, $\mu^* > \gamma$. Notice that γ is in N since ${}^{\omega}N \subseteq N$ and each γ_m is in N. Then for every $m < \omega$, $\gamma_m = \mathbb{C}(\overline{\kappa}, \beta(\mu_{mn}), \beta(\mu))(\mu) < \mu^*$. So the case (ii) of the condition (e) (for N) should hold. In particular, taking $\beta = \beta^*(\mu)$ there, we will have unboundedly many in μ indiscernibles τ_1 for $\overline{\kappa}$ such that $\beta^*(\mu) < \beta(\tau_1) < \beta(\mu)$. Pick some such τ_1 above μ_{on}. Let $m < \omega$ be so that $\mu_{mn} > \tau_1$. By the assumptions on d_n, then $\beta(\mu_{mn}) > \beta(\tau_1)$. Hence $\beta(\mu_{mn}) > \beta^*(\mu)$. Contradiction. ∎

LEMMA 2.9: *Suppose that* cf $\alpha^* = \kappa^+$. *For every $n < \omega$ there exists a final segment d'_n of limit points of d_n so that for every $\mu, \mu' \in d'_n$, $\beta^*(\mu) = \beta^*(\mu')$.*

Proof: Suppose otherwise. Then for every limit $\mu \in d_n$ there is a limit $\mu' \in d_n - \mu$ with $\beta^*(\mu') > \beta^*(\mu)$. But then also $\beta^*(\mu') > \beta(\mu)$. Since otherwise, by Lemma 2.8, $\beta(\mu) > \beta^*(\mu') > \beta^*(\mu)$ which is impossible.

Define an increasing sequence $\langle \mu_m \mid m < \omega \rangle$ of limit elements of d_n so that $\beta^*(\mu_{m+1}) > \beta^*(\mu_m)$. Let $\mu = \bigcup_{m<\omega} \mu_m$. But then, by Lemma 2.8,

$$\beta^*(\mu) < \bigcup_{m<\omega} \beta(\mu_m) = \bigcup_{m<\omega} \beta^*(\mu_m) \leq \beta(\mu)$$

which contradicts the definition of $\beta^*(\mu)$. ∎

So, inside every d_n, $\beta^*(\mu)$ is stabilized under the assumption of cf $\alpha^* = \kappa^+$. Let us remove this assumption now. So suppose cf $\alpha^* = \omega$. Pick a cofinal sequence $\langle \alpha_k^* \mid k < \omega \rangle$ to α^* consisting of ordinals of cofinality $> \aleph_0$ so that for every $k<\omega$, $\beta<\alpha_k^*$ there unboundedly many in κ indiscernible for measures in the interval (β,α_k^*). Since $\operatorname{cf}\alpha^*=\aleph_0$, the choice of α^* insures that it is possible. For every $k<\omega$ pick a set $A_k^*\in\mathcal{F}(\kappa,\alpha_k^*)$ which does not belong to any measure above α_k^*. Assume that for every $\delta\in B$ $\delta^*=\bigcup_{k<\omega}\delta_k^*$, where δ_k^* is the maximal measure s.t. $\delta\cap A_k^*\in\mathcal{F}(\delta,\delta_k^*)$. For $\mu\in C$, define $\langle\beta_k^*(\mu)\mid k<\omega\rangle$ in the obvious way.

Then the following analogs of Lemmas 2.8 and 2.9 hold.

LEMMA 2.8.1: *For every $k,n<\omega$, for every increasing sequence $\langle\mu_{mn}\mid m<\omega\rangle$ of elements of d_n*

$$\beta_k^*\left(\bigcup_{m<\omega}\mu_{mn}\right)<\bigcup_{m<\omega}\beta(\mu_{mn})\le\beta\left(\bigcup_{m<\omega}\mu_{mn}\right)$$

LEMMA 2.9.1: *For every $k,n<\omega$ there exists a final segment $d_n^{(k)}$ of limit points of d_n so that for every $\mu,\mu'\in d_n^{(k)}$, $\beta_k^*(\mu)=\beta_k^*(\mu')$.*

But now using Lemma 2.9.1 we can stabilize step by step all $\beta_k^*(\mu)$ $(k<\omega)$. So the following holds.

LEMMA 2.9.2: *For every $n<\omega$ there exists a final segment d_n' of limit points of d_n so that for every $\mu,\mu'\in d_n'$, $\beta^*(\mu)=\beta^*(\mu')$.*

Let us identify, from now on, d_n' with d_n for simplification of the notation. Denote also $\beta^*(\mu)$ for $\mu\in d_n$ by β_n^*.

Notice that the arguments above work still if we replace $\bar\kappa$ by any $\delta\in d_n$ $(n<\omega)$ with $\operatorname{cf}\delta\ge\omega_1$. Just instead of dealing with measures over $\bar\kappa$ deal with measures over δ.

Actually, if we restrict ourselves to δ's below $\bar\kappa$, then the assumptions that N is good and it is of cardinality $2^{\aleph_0}$ are not used.

Define, in V, $\lambda=\kappa^+$ if κ is an inaccessible and $\lambda=\mu$ if $\kappa=\mu^+$. Let g_λ be a function which uniformly represents λ. Let $A\in\bigcap\{\mathcal{F}(\kappa,\beta)\mid\alpha^*\le\beta<\alpha^*+\lambda\}$. We like to find $\beta<\alpha^*$ so that $A\in\bigcap\{\mathcal{F}(\kappa,\beta')\mid\beta\le\beta'<\beta+\lambda\}$.

Set $B(A)=\{\delta<\kappa\mid$ there exists $\delta^*<O^{\vec{\mathcal{F}}}(\delta)$ such that (1) it is largest s.t. $A^*\cap\delta\in\mathcal{F}(\delta,\delta^*)$, (2) $A\cap\delta\in\bigcap\{\mathcal{F}(\delta,\beta)\mid\delta^*\le\beta<\delta^*+g_{\lambda(\delta)}$. Intuitively, $B(A)$ takes care of relevant measures which maybe are above $\alpha^*+\lambda$. Then $A\cup B(A)\in\bigcap\{\mathcal{F}(\kappa,\beta)\mid\beta$ is a relevant ordinal$\}$.

Let us assume that points of the club C of cofinality ω are in $(A^*\cup B)\cap(A\cup B(A))$.

It was shown above, that for the model N, $(\beta_n^*(\mu))^N$'s are stabilized. Our aim now will be to find ordinals β_n^* which do not depend on a particular model. The models we shall consider below will satisfy the conditions (a), (c)–(e) of the definition of N. So further, by a model we will mean a model satisfying (a), (c)–(e). Unions of elementary chains of good models will be used. We do not know whether such models are good. But what we had insured by picking the set Y to be in the generic filter is that these models are contained in good ones. For a submodel N_1 of a good model N_0 and for an indiscernible $\mu \in N_0 \cap N_1$ let us denote by $\beta^{N_0}(\mu), \beta^{N_1}(\mu)$. Notice that, by Mitchell [Mi4, Lemma 1.5], $\beta^{N_0}(-)$ does not depend on particular good $N_1 \supseteq N_0$, at least a final segment of indiscernibles. The lemmas above are valid for closed under ω-sequences submodels of good models. Also for some nonclosed submodels N' of a good model $d_n^{N'}$, ${\beta_n^*}^{N'}$ still may exist.

Further, by a model we shall mean a fine submodel of a good model. Assume also for simplification of the notions that $\overline{\kappa}^N = \kappa$.

LEMMA 2.10.1: *For every N' there exists $N \supset N'$ so that, for every $N'' \supseteq N$, ${\beta_n^*}^{N''} \leq {\beta_n^*}^{N}$ for all but finitely many n's.*

Proof: Suppose otherwise. Define an elementary chain $\langle N_i \mid i < \omega_1 \rangle$ so that

(1) N_0 is a counterexample,

(2) ${}^{\omega}N_{i+1} \supseteq N_{i+1}$,

(3) $h^{N_{i+1}} \in N_{i+2}$,

(4) $N_i = \bigcup_{j<i} N_j$ for a limit i,

(5) for infinitely many n's ${\beta_n^*}^{N_{i+1}} > {\beta_n^*}^{N_i}$, if ${\beta_n^*}^{N_i}$'s exist,

(6) $N_{i+1} \supset N_i$,

(7) $\langle \bigcup (N_i \cap \tau_n) \mid n < \omega \rangle \in N_{i+1}$.

Set $N = \bigcup_{i<\omega_1} N_i$. ${}^{\omega}N \subset N$, so $(\beta_n^*)^N$'s exist. For every $i < \omega_1$ there is $n(i) < \omega$ such that all indiscernibles of N_i for κ which are above $\tau_{n(i)}$ are indiscernibles of N for the same measures, since there exists a good model containing N and N_i. Find a stationary $S \subseteq \omega_1$ and $n_0 < \omega$ such that for every $i \in S$, $n(i) = n_0$. Suppose for simplicity that $n_0 = 0$. Since each d_n^N is an increasing continuous union of $\langle N_i \cap d_n^N \mid i < \omega_1 \rangle$, ${\beta_n^*}^{N_i}$ and $d_n^{N_i}$ are defined and $d_n^{N_i}$ is almost contained in d_n^N. Then ${\beta_n^*}^{N_i} = {\beta_n^*}^{N}$. $\langle \eta_n \mid n < \omega \rangle \in N_{i+0}$. Consider N_{i_0+1}. Find $n_1 < \omega$ so that all indiscernibles of N_{i_0+1} for κ which are above τ_{n_1} are indiscernibles of N for the same measures. Fix some $n < \omega$ above n_1

such that ${\beta_n^*}^{N_{i_0+1}} > {\beta_n^*}^{N_{i_0}} = {\beta_n^*}^N$. If some $\mu \in d_n^{N_{i_0}+1}$ is above η_n then $\beta^N(\mu)$ should be bigger than $\beta^N(\eta_n)$, since $\left(\beta^N(\mu)\right)^* = {\beta^*}^{N_{i_0+1}} > {\beta_n^*}^N$. But then, by the definition of d_n, $\left(\beta^N(\mu)\right)^* = {\beta_n^*}^N$, which is impossible. So $\sup d_n^{i_0+1} = \eta_n$. But $\sup d_n^{N_{i_0}+1} = \sup(N_{i_0+1} \cap \tau_n) > \eta$ since $\eta_n \in N_{i_0+1}$. Contradiction. ∎

LEMMA 2.10.2: *There exists a sequence $\langle \beta_n^* \mid n < \omega \rangle$ so that for every N' there is $N \supseteq N'$ as in the conclusion of Lemma 2.10.1, so that $\beta_n^* = {\beta_n^*}^N$ for all but finitely many n's.*

Proof: It is enough to prove the following:

CLAIM: *For every N' there exists $N \supset N'$ as in the conclusion of Lemma 2.10.1 so that for every $N'' \supseteq N$ there is $N''' \supseteq N''$ satisfying ${\beta_n^*}^{N'''} = {\beta_n^*}^N$ for all but finitely many n's.*

Proof: Suppose otherwise. Let N_0 be a counterexample. For a model $N \supseteq N_0$ denote by $Z(N)$ a model $N'' \supseteq N$ such that for every $N''' \supseteq N''$, ${\beta_n^*}^{N'''} \neq {\beta^*}^N$ for infinitely many n's. Define an elementary chain $\langle N_i \mid i < \omega_1 \rangle$ as follows:

(1) $N_i = \bigcup_{j<i} N_j$ for a limit i,
(2) ${}^\omega N_{i+3} \subseteq N_{i+3}$ is a good model,
(3) $h^{N_{i+3}} \in N_{i+4}$,
(4) for a limit i, $N_{i+1} \supseteq N_i$ is given by Lemma 2.10.1,
(5) for a limit i, $N_{i+2} = Z(N_{i+1})$.

Set $N = \bigcup_{i<\omega_1} N_i$. As in Lemma 2.10.1, $(\beta_n^*)^{N_i}$ exists and ${\beta_n^*}^{N_i} = {\beta_n^*}^N$ for i in a stationary set $S \subseteq \omega_1$.

For $i < \omega_1$ denote $\langle {\beta_n^*}^{N_i} \mid n < \omega \rangle$ by $\overline{\beta}^i$ if it is defined. Let $\overline{\beta}^i < \overline{\beta}^j$ mean that, for infinitely many n's, ${\beta_n^*}^{N_i} < {\beta_n^*}^{N_j}$. Then $\overline{\beta}^1 > \overline{\beta}^{\omega+1} > \overline{\beta}^{\omega+\omega+1} > \cdots > \overline{\beta}^{i+1} > \cdots$, for a limit $i < \omega_1$. By the proof of Lemma 2.10.1, for $i \in S$ big enough $\overline{\beta}^i \not< \overline{\beta}^{i+1}$. By the choice of N_{i+1} then ${\beta^*}^{N_i} = {\beta_n^*}^{N_{i+1}}$ for all but finitely many n's. But since ${\beta^*}^{N_i} = {\beta_n^*}^N = {\beta_n^*}^{N_j}$ for all but finitely many n's, where $i, j \in S$ are big enough, $\overline{\beta}^{i+1} \not> \overline{\beta}^{j+1}$. Contradiction. ∎

Remark: It is easy to make N in Lemma 2.10.2 closed under ω-sequences. Just construct an elementary chain $\langle N_i \mid i < \omega_1 \rangle$ taking N_{i+1} to be as in Lemma 2.10.2, ${}^\omega N_{i+2} \subseteq N_{i+2}$ for a limit i. $N = \bigcup_{i<\omega_1} N_i$ will be as required.

We split the proof now into two cases: (1) κ is accessible; (2) κ is inaccessible.

Suppose first that $\kappa = \lambda^+$ for λ which is a successor of a regular cardinal of cofinality $> \omega$. Let λ^- denote the predecessor of λ, i.e. $\lambda = (\lambda^-)^+$. Suppose $\lambda^- = (\lambda^-)^\omega$.

PROPOSITION 2.10: *α^* is a λ-repeat point.*

Proof: Let us deal, for simplicity, with $\kappa = \aleph_3$ and $\lambda = \aleph_2$.

We use M, N, $\langle \tau_n^N \mid n < \omega \rangle$, $\langle d_n^N \mid n < \omega \rangle$, $\langle \beta_n^* \mid n < \omega \rangle$ etc. defined above, only suppose that $A, B(A)$ are also in N. Continue to work inside M. Suppose also that $\min d_0^N$ is above the supports of $A, A^*, B, B(A)$.

If for some $n < \omega$

$$A \in \bigcap \{ \mathcal{F}(\kappa, \alpha) \mid \beta_n^* \leq \alpha < \beta_n^* + \omega_2 \}$$

then we are done. Suppose otherwise.

Define then γ_n to be the least ordinal between β_n^* and $\beta_n^* + \omega_2$ such that $A \notin \mathcal{F}(\kappa, \gamma_n)$. Without loss of generality assume that $A \notin \mathcal{F}(\kappa, \gamma_n)$ implies that also $A \notin \mathcal{F}(\kappa, \gamma)$ for every γ, $\gamma_n \leq \gamma < \omega_2$. It was shown that $\gamma_n > \beta_n^*$ for all but finitely many n's. Actually, it can be used to show that $\gamma_n \geq \beta_n^* + \omega_1$ since N, a model of Lemma 2.10.2, can be chosen to be closed under ω-sequences.

Set $\delta_n = \sup\{\beta(\tau) \mid \tau \in (C \cap C_h \cap \tau_{n+1}) - \min d_n^N$, τ, is an indiscernible of N for $\kappa\}$ for every $n < \omega$. W. l. of g. assume that

$$\delta_n = \sup\{\beta(\tau) \mid \tau \in (C \cap C_h \cap \tau_{n+1}) - \xi\} \quad \text{for every } \xi \in [\min d_n, \tau_{n+1}).$$

LEMMA 2.11: *For every $\delta < \aleph_2$ for every N' there exists $N \supseteq N'$ so that ${\beta_n^*}^N = \beta_n^*$ and $\delta_n^N \geq \beta_n^* + \delta$ for all but finitely many n's.*

Proof: Suppose otherwise. Pick δ to be the minimal counter-example. Let N' be so that for every $N \supseteq N'$ with ${\beta_n^*}^N = \beta_n^*$ for all but finitely many n's, $\delta_n^N < {\beta_n^*}^N + \delta$ for all but finitely many n's. Pick $\langle \delta_i \mid i < \omega_i \rangle$ to be a cofinal sequence to δ. If δ is a successor ordinal then let $\delta_i = \delta - 1$ for every i. Define now an elementary chain $\langle N_i \mid i < \omega_1 \rangle$ satisfying the following conditions:

(1) $N_0 = N'$,
(2) ${}^\omega N_{i+1} \subseteq N_{i+1}$,
(3) $h^{N_{i+1}} \in N_{i+2}$,
(4) $N_i = \bigcup_{j<i} N_j$ for a limit i,
(5) ${\beta_n^*}^{N_{i+1}} = \beta_n^*$ for all but finitely many n's,

(6) $N_{i+1} \supseteq N_i$,

(7) $\langle \bigcup (N_i \cap \tau) \mid n < \omega \rangle \in N_{i+1}$,

(8) $\delta_n^{N_{i+1}} \geq \beta_n^* + \delta_i$ for all but finitely many n's.

Set $N = \bigcup_{i<\omega_1} N_i$. Find a stationary $S \subseteq \omega_1$ and $n_0 < \omega$ so that for every $i \in S$ all indiscernibles of N_i for κ above τ_{n_0} are indiscernibles for N for the same measures. As in Lemma 2.10.1, then ${\beta_n^*}^{N_i} = {\beta_n^*}^N$ for $n > n_0$, $i \in S$. Pick $i_0 \in S$ so that $\langle \eta_n \mid n < \omega \rangle \in N_{i_0}$, where $\eta_n = \min d_n$. As in Lemma 2.10.1 for every $i \geq i_0$ there only finitely many n's such that ${\beta_n^*}^{N_i} > {\beta_n^*}^N$. But ${\beta^*}^{N_{i+1}} = \beta_n^*$ for all but finitely many n's. Hence the maximality of β_n^*'s implies that $\beta_n^* = {\beta_n^*}^{N_{i+1}} = {\beta_n^*}^N$ for all but finitely many n's.

Pick now a set $S_1 \subseteq \{i+1 \mid i \in S - i_0\}$ of cardinality ω_1 and $n_1, n_0 < n_1 < \omega$ so that for every $i \in S_1$ all indiscernibles of N_i for κ above τ_{n_1} are indiscernibles of N for the same measures. Then for all $n > n_1$, $i+1 \in S_1$ big enough s.t. $\langle \delta_n^N \mid n < \omega \rangle \in N_{i+1}$ $\delta_n^N \geq \beta_n^* + \delta_i$. Hence $\delta_n^N \geq \beta_n^* + \delta$ for every $n > n_1$. Contradiction. ∎

It is easy now to finish the proof of the proposition. Let $\gamma = \bigcup_n \gamma_n$. Then $\gamma < \omega_2$ and so, by Lemma 2.11, there exists N satisfying $\delta_n^N > {\beta_n^*}^N + \gamma$ for all but finitely many n's. Find an indiscernible $\delta \in N \cap C$ for κ with $\beta^N(\delta) \geq {\beta_n^*}^N + \gamma$, where n is such that τ_{n-1} is above the support of $A, B(A)$. Let $N^* \supseteq N$ be a good model which defines $\beta^N(-)$. Then $\mathcal{F}^{N^*}(\kappa, \beta^N(\delta))$ is the real $\mathcal{F}(\kappa, \beta^N(\delta))$. Also $A \cup B(A) \in \mathcal{F}(\kappa, {\beta_n^*}^N + \gamma)$. Contradiction. ∎

Remark: The property $(\lambda^-)^\omega = \lambda^-$ (where $(\lambda^-)^{++} = \kappa$) was used in the above proof in order to apply the Mitchell Covering Lemma where the submodels are assumed to be closed under ω-sequences. If it is possible to remove this assumption from the Mitchell Covering Lemma, then the above construction can be applied also for $\kappa = \lambda^{++}$ with $\operatorname{cf} \lambda = \omega$ to produce a λ^+-repeat point.

Suppose now that κ is a weakly inaccessible so that $\lambda^\omega < \kappa$ for every $\lambda < \kappa$

PROPOSITION 2.12: *α^* is a $< \kappa$-repeat point.*

Proof: Let $\gamma < \kappa$ be a regular cardinal. We would like to show that α^* is a γ-repeat point. If there exists a good model of cardinality γ then the application of Lemma 2.9.2 to this model gives the desired statement. In general we know that there are good models of cardinality $2^{\aleph_0}$. So let us pick a submodel N of H_{χ^+} (χ large enough) for cardinality γ^+ containing $\langle \tau_n \mid n < \omega \rangle$, C, A^* etc. Using Lemma 2.9.2, find indiscernibles of N for κ $\langle \mu_n \mid n < \omega \rangle$ so that $\tau_{n+1} > \mu_n > \tau_n$,

$\mu_n \in C$, cf $\mu_n = \omega$ and $\beta^N(\mu_n) \geq \beta^{*^N}(\mu_n)+\gamma$. Let N_0 be a elementary submodel of N of cardinality $2^{\aleph_0}$ containing $\langle \mu_n \mid n < \omega\rangle\gamma$, C, A^* etc. By Mitchell [Mi4], all but finitely many of μ_n's remain indiscernibles for κ in N_0. Let n_0 be so that $\tau_{n_0} > \gamma$ and for every $n > n_0$, μ_n is an indiscernible for κ in N_0.

The set $A_\gamma = \{\delta < \kappa \mid$ there exists the maximal $\delta^* < O^{\mathcal{F}}(\delta)$ s.t. $A^* \cap \delta \in \mathcal{F}(\delta, \delta^*)$ and $O^{\mathcal{F}}(\delta) \geq \delta^* + \gamma\}$ is definable from γ and A^* in $\mathcal{K}(\mathcal{F})$. So its support is below τ_{n_0} in both N and N_0. Let n be above n_0. Then $A_\gamma \in \mathcal{F}^N(\kappa, \beta^N(\mu_n))$ and, hence, $\mu_n \in A_\gamma$. But now, in N_0, $\mu_n \in A_\gamma$ and it implies that $A_\gamma \in \mathcal{F}^{N_0}(\kappa, \beta^{N_0}(\mu_n))$. So $\beta^{*^{N_0}}(\mu_n)+\gamma \leq \beta^{N_0}(\mu_n)$. And this is true for every $n \geq n_0$. The continuation is as at the end of the proof of Proposition 2.1. ∎

The same argument applies to $\kappa = \lambda^{++}$ with cf $\lambda = \omega$ and $\mu^\omega < \lambda$ for $\mu < \lambda$. So the following holds.

PROPOSITION 2.13: *Let $\kappa = \lambda^{++}$, cf $\lambda = \omega$ and $\mu^\omega < \lambda$ for every $\mu < \lambda$. Then α^* is $< \lambda$-repeat point.*

Suppose now that $\kappa = \lambda^+$ for weakly inaccessible λ so that $\mu^\omega < \lambda$ for every $\mu < \lambda$. The arguments above show that then α^* is a $< \lambda$-repeat point.

If every submodel of cardinality $< \lambda$ is contained in a good submodel of cardinality $< \lambda$, then an easy modification of these arguments gives that α^* is a λ-repeat point. But we do not know whether this is the case. Below, we shall obtain the same conclusion restricting ourselves to good models of cardinality $2^{\aleph_0}$.

PROPOSITION 2.14: *If $\kappa = \lambda^+$ for a weakly inaccessible λ s.t. $\mu^\omega < \lambda$ for every $\mu < \lambda$, then α^* is a λ-repeat point.*

Proof: Suppose otherwise. Without loss of generality suppose that the set A (picked in the beginning) witnesses this, i.e., for every $\beta < \alpha^*$ there is some $\rho < \lambda$ so that $A \notin \mathcal{F}(\kappa, \beta+\rho)$. Also all τ_n's may be assumed to be of cofinality λ. Since otherwise, pick a submodel of cardinality $< \lambda$ which contains cofinal sequences to all of τ_n's of cardinality less than λ. Then, as in Lemma 2.1, construct new τ_n's. If they are still of cofinality less than λ, then continue the process. It should be stabilized in less than $(2^{\aleph_0})^+$ steps. The stabilized τ_n's will be of cofinality λ.

LEMMA 2.15: *There exists a submodel N' of cardinality $2^{\aleph_0}$ so that for every $\rho < \lambda$ there are $N'' \supseteq N'$ of the same cardinality, $n_0 < \omega$ and the sequence of indiscernibles of $N''$for κ in C $\langle \mu_n \mid n \geq n_0\rangle$ so that for every $n \geq n_0$,*

$\beta^{N''}(\mu_n) - \beta^{*^{N''}}(\mu_n) \geq \rho$ and $\beta^{*^{N''}}(\mu_n) = \beta_n^*$, where $\langle \beta_n^* \mid n < \omega \rangle$ is defined as in Lemma 2.10.2.

The proposition follows easily from the lemma. Just otherwise pick for every $n \geq n_0$, $\rho_n < \lambda$, so that $A \notin \mathcal{F}(\kappa, \beta_n^* + \rho_n)$. Set $\rho = \bigcup_n \rho_n$. Since λ is a regular, $\rho < \lambda$. Pick now N'', $\langle \mu_n | n \geq n_0 \rangle$ as in the statement of the lemma for this particular ρ. Then for all n's big enough $\mu_n \in A$. It implies that $A \in \mathcal{F}(\kappa, \beta^{N''}(\mu_n))$. But $\beta^{N''}(\mu_n) - \beta_n^* \geq \rho$. So A also belongs to $\mathcal{F}(\kappa, \beta_n^* + \rho)$. Contradiction.

Proof of the lemma: Suppose otherwise. Then for every N' of cardinality $2^{\aleph_0}$ there is $\rho(N') < \lambda$ so that for every $N'' \supseteq N'$ of the same cardinality, every sequence of indiscernibles of N'' of κ in C $\langle \mu_n \mid n < \omega \rangle$, so that $\beta^{N''}(\mu_n) = \beta_n^*$, $\beta^{N''}(\mu_n) - \beta_n^* < \rho(N')$ for all but finitely many n's.

Let $N', \rho(N')$ be as above. Denote $\rho(N')$ simply by ρ. Define by induction an increasing continuous sequence of submodels $\langle N_i \mid i < \rho^+ \rangle$ so that

(1) $N = N'$,

(2) $|N_i| < \lambda$ for every $i < \rho^+$,

(3) for every i, N_{i+1} contains a submodel N_i' of cardinality $2^{\aleph_0}$ so that $\{\bigcup(N_i \cap \tau_n) \mid n < \omega\} \subseteq N_i'$ and N_i' has a sequence $\langle \mu_n \mid n < \omega \rangle$ of indiscernibles for κ in C so that for all but finitely many n's, $\mu_n \in \tau_{n+1} \setminus \bigcup(N_i \cap \tau_{n+1})$ and $\beta^{*^{N_i'}}(\mu_n) = \beta_n^*$.

Set $N = \bigcup\{N_i \mid i < \rho^+\}$. Consider the sequence $\langle d_n^N \mid n < \omega \rangle$ defined as in Proposition 2.10. Condition (3) insures that $|d_n^N| = \rho^+$. So, for every n, there exists $\eta_n \in d_n^N$ such that $\beta^N(\mu) - \beta^{*^N}(\mu) \geq \rho$ for every μ, $\mu \in C \cap N \cap \tau_{n+1} - \eta_n$ and $\beta^N(\mu) \geq \beta^N(\eta_n)$. Let us define now a submodel N^* of N of cardinality $2^{\aleph_0}$. It will be the union of the increasing continuous chain $\langle N_i^* \mid i < \omega_1 \rangle$, which is defined as follows. Let k_0 be the least so that $\{\eta_n \mid n < \omega\} \subseteq N_{k_0}$. Set N_0^* to be a submodel of N_{k_0} of cardinality $2^{\aleph_0}$ containing $\{\eta_n \mid n < \omega\} \cup N_0$. Let $\eta_n^0 = \eta_n$ for every $n < \omega$. For a limit i, let N_i^* be $\bigcup_{j<i} N_j^*$. Set $k_i = \bigcup_{j<i} k_j$ and $\eta_n^i = \bigcup_{j<i} \eta_n^j$. So $N_i^* \subseteq N_{k_i}$. Suppose that $\langle N_j^* \mid j \leq i \rangle$, $\langle k_j \mid j \leq i \rangle$ are defined. Define N_{i+1}. For every $n < \omega$, let η_n^{i+1} be the minimal $\eta \in d_n^N - \bigcup(\tau_{n+1} \cap N'_{k+i})$ so that $\beta^N(\eta) > \beta^N(\eta_n^i)$. Let k_{i+1} be the least $j > k_i$ so that $N_j \supseteq \{\eta_n^{i+1} \mid n < \omega\}$. Pick N_{i+1}^* to be a submodel of $N_{k_{i+1}}$ of cardinality $2^{\aleph_0}$ containing $N_i^* \cup N'_{k_i} \cup \{\eta_n^{i+1} \mid n < \omega\}$.

Clearly, $\{\eta_n^i \mid i < \omega_1\} \subseteq d_n^N$ for every n. Also $\{\eta_n^i \mid i < \omega_1\}$ is a club in

$\bigcup(N^* \cap \tau_{n+1})$. So some subclub t_n of it is contained in $d_n^{N^*}$. Since $\beta^N(\eta_n^i) \geq \beta^N(\eta_n)$, $\beta^N(\eta_n^i) - \beta^{*^N}(\eta_n^i) \geq \rho$ for every $i < \omega_1$. Consider the set $A_\rho = \{\delta < \kappa \mid$ there exists a maximal $\delta^* < O^{\mathcal{F}}(\delta)$ so that $A^* \cap \delta \in \mathcal{F}(\delta, \delta^*)$ and this δ^* satisfies $\delta^* + \rho \leq O^{\mathcal{F}}(\delta)\}$. It is determinable from A^* and ρ. So starting with some $n_0 < \omega$, for every $\mu \in t_n$, $\mu \in A_\rho$ in N and, hence, in N^*. So $\beta^{N^*}(\mu) > \beta^{*^{N^*}}(\mu) \geq 0$. The choice of ρ implies that starting with some $n_1 \geq n_0$, $\beta^{*^{N^*}}(\mu) \neq \beta_n^*$ for every $\mu \in t_n$. By the choice of β_n^*'s, then $\beta^{*^{N^*}}(\mu) < \beta_n^*$ for every $n \geq n_2 \geq n_1$. Denote $\min t_n$ by ξ_n. Pick $i < \omega_1$ so that $\{\xi_n \mid n < \omega\} \subseteq N_i^*$. Consider N_i' and N^*. They are submodels of a good model. So, starting with some $n_3 \geq n_2$ all the indiscernibles of N' for κ above τ_{n_3} are indiscernibles of N^* for the same measures. By the choice of N_i', for every $n \geq n_3$, there exists $\mu_n \in \tau_{n+1} \setminus \xi_n$ so that $\beta^{*^{N_1'}}(\mu_n) = \beta_n^*$. Then, $\beta^{*^{N^*}}(\mu_n) = \beta_n^* > \beta^{*^{N^*}}(\xi_n)$. So $\beta^{N^*}(\mu_n) > \beta^{N^*}(\xi_n)$. By the choice of $d_n^{N^*}$, this implies $\beta^{*^{N^*}}(\mu_n) = \beta^{*^{N^*}}(\xi_n)$. Contradiction. ∎

Let us show now that unless there exists an $(\omega, \kappa^+ + \kappa^+)$-repeat point cf α^* should be ω. Suppose otherwise. Then, as it was shown after Lemma 2.7, there exists the maximal reflecting set $\{\beta_0, \ldots, \beta_{n-1}\}$. Let us assume for simplicity that $n \leq 1$. Denote β_0 by α^{**} if $n = 1$ and let $\alpha^{**} = 0$ if $n = 0$.

Until now we have dealt with one club C. In order to get the contradiction we shall use some different clubs. But first let us prove the following.

Lemma 2.16: *There exists N so that for every $N' \supseteq N$ the set of indiscernibles for κ, $\tau \in C$, cf $\tau = \aleph_0$ with $\beta^{*N'}(\tau) > \bigcup\{\beta^N(\gamma) \mid \gamma \in C$ is an indiscernible for κ in $N\}$ is bounded in κ.*

Proof: Suppose otherwise. Define a chain $\langle N_i \mid i < \omega_1 \rangle$ so that

(1) $N_{i+1} \supseteq N_i \cup \{h_i^{N_i}\}$,
(2) $N_i = \bigcup_{j<i} N_j$ for limit i,
(3) there are unboundedly many in κ indiscernibles $\tau \in C$ for N_{i+1}, cf $\tau = \aleph_0$ with

$$\beta^{*N_{i+1}}(\tau) > \bigcup\{\beta^{N_j}(\gamma) \mid \gamma \in C \text{ is an indiscernible for } \kappa \text{ in } N_j\} .$$

Set $N = \bigcup_{i<\omega_1} N_i$. Find $S \subseteq \omega_1$ for cardinality ω_1 consisting of successor ordinals and $n_0 < \omega$ such that for every $i \in S$, for every $n \geq n_0$ and for every indiscernible $\gamma \in N_i - \tau_{n_0}$ for κ $\beta^{N_i}(\gamma) = \beta^N(\gamma)$, also there is $m \geq n$ and an indiscernible $\gamma_m^i \in [\tau_m, \tau_{m+1})$ for κ in N_i satisfying the condition (3) above.

Suppose for simplicity, that for every $m \geq n_0$, γ^i_m is defined for ω_1 i's in S. Otherwise just deal with intervals $[\tau_m, \tau_{m+1})$ for m's satisfying this.

Set $b_m = \{\gamma^i_m \mid i \in S,\ \gamma^i_m \text{ is defined}\}$ and $\gamma_m = \bigcup b_m$.

CLAIM: *There is $n_1 \geq n_0$ so that for every $m \geq n_1$, for every indiscernible $\delta_m \in C \cap [\tau_m, \tau_{m+1})$ for κ there is $\gamma \in b_m$ with $\beta^{*N}(\gamma) > \beta^N(\delta_m)$.*

Proof: Suppose otherwise. Then for every $n \geq n_0$ there are $m_n \geq n$ and indiscernible $\delta_{m_n} \in C \cup [\tau_{m_n}, \tau_{m_n+1})$ such that $\beta^N(\delta_{m_n}) \geq \beta^{*N}(\gamma)$ for every $\gamma \in b_{m_n}$. There exists $i_0 \in S$ such that $\{\delta_{m_n} \mid n_0 \leq n < \omega\} \subseteq N_{i_0}$. But it implies $\beta^{N_{i_0}}(\delta_{m_n}) = \beta^N(\delta_{m_n})$ for every $n \geq n_0$. Let i_1 be any element of $S - i_0$. Then $\beta^{*N}(\gamma^{i_1}_{m_n}) = \beta^{*N_{i_1}}(\gamma^{i_1}_{m_n}) > \beta^{N_{i_1-1}}(\delta_{m_n}) = \beta^{N_{i_0}}(\delta_{m_n}) = \beta^N(\delta_{m_n})$ for every n s.t. $\gamma^{i_1}_{m_n}$ is defined and $N_{i_0}, N_{i_1} N$ agree about the function $\beta(\cdot)$. Contradiction. ∎

Let $m \geq n_1$. Since cf $\gamma_m = \omega_1$, there is a club $e_m \subseteq \gamma_m$ consisting of elements of C which are indiscernibles for κ so that for every $\tau, \tau' \in e_m$ and τ'', $\tau < \tau'' < \tau'$, $\beta^N(\tau) < \beta^N(\tau')$ and $\beta^N(\tau'') < \beta^N(\tau')$. Define now an increasing sequence $\langle t_n \mid n < \omega \rangle$ of indiscernibles in $C \cap \gamma_m$ as follows. Set $t_0 = \min e_m$. Let $t_{n+1} = \min\{\gamma \in e_m \mid \beta^{*N}(\gamma) > \beta^N(t_n)\}$. Such t_{n+1} exists by the claim.

Then $\beta^{*N}(\bigcup_{n<\omega} t_n) \geq \bigcup_{n<\omega} \beta^N(t_n)$, which is impossible by Lemma 2.8. Contradiction. ∎

Set $\alpha(C) = \bigcup\{\beta^N(\gamma) \mid \gamma \in C \text{ is indiscernible for } \kappa \text{ in } N\}$, where N is as in Lemma 2.16.

Since $|N| < |\kappa|$ and $\mathrm{cf}^{\mathcal{K}(\vec{\mathcal{F}})} \alpha^* = \kappa^+$ $\alpha(C) < \alpha^*$. Also $\alpha(C) < \alpha^{**}$, if $\alpha(C) \leq \alpha^{**}$. Pick a set $A_1 \in \mathcal{F}(\kappa, \alpha^*)$ so that $A_1 \subseteq A^*$ and $A_1 \notin \bigcup\{\mathcal{F}(\kappa,\gamma) \mid \alpha^{**} \leq \gamma \leq \alpha(C)$ if $\alpha^{**} < \alpha(C)$ and $0 \leq \gamma \leq \alpha(C)$ otherwise$\}$.

Let B_1 be defined as B but only replace A^* by A_1. Pick a club $C_1 \in V$ which points of cofinality ω are in $A_1 \cup B_1$.

Let us now consider separately two cases: (1) $\alpha^{**} = 0$; (2) $\alpha^{**} > 0$.

CASE 1: $\alpha^{**} = 0$. Let N be as in Lemma 2.16. Pick $N' \supseteq N$ so that $A_1, B_1, C_1 \in N'$. Since $C \cap C_1$ is a club in V, there are unboundedly many indiscernibles $\tau \in C \cap C_1$ for κ in N' of cofinality ω. By the choice of A_1, B_1, C_1, for sufficiently large $\tau \in C \cap C_1$, $\beta^*(\tau) > \alpha(C)$. But by Lemma 2.17.1, it is possible only for a bounded in κ number of τ's. So $C \cap C_1$ is bounded in κ, which is impossible. Contradiction.

CASE 2: $\alpha^{**} > 0$. The arguments of Case 1 imply that there are only boundedly many in κ indiscernibles τ in $C \cap C_1$ for κ in N' with $\beta(\tau) \geq \alpha^{**}$. Consider now $C \cap C_1$ instead of C. Then $\alpha(C \cap C_1) < \alpha^{**}$. Pick a set $A_2 \in \mathcal{F}(\kappa, \alpha^*)$ so that $A_2 \subseteq A_1$ and $A_2 \notin \bigcup\{\mathcal{F}(\kappa,\gamma) \mid \gamma \leq \alpha(C \cup C_1)\}$. Find B_2 and C_2 for A_2 as B_1 and C_1 for A_1. Apply the argument of Case 1 to $C \cap C_1$ and C_2. It will imply boundedness of $C \cap C_1 \cap C_2$, which again leads to the contradiction.

So cf α^* should be ω.

This completes the proof of Theorem 2.5.1. ∎

Note that, if the assumption ${}^{\omega}N \subseteq N$ can be removed from the Mitchell covering lemma, then the conclusion of (e) of 2.5.1 can be improved to an (ω, λ^+)-repeat point.

It is possible to replace $\aleph_0$ in 2.5.1 by any regular $\mu < \kappa$ so that $\mu^+ < \kappa$. Note that by Jech–Magidor–Mitchell–Prikry [J-M-M-P] precipitousness of $\mathrm{NS}^{\mu}_{\mu^+}$ is equiconsistent with a one measurable.

THEOREM 2.17.0: *Suppose that N^{μ}_{κ} is precipitous for a regular uncountable μ with $\mu^+ < \kappa$, then the conclusion of* 2.5.1 *holds.*

Sketch of the proof: The only difference in $\mathrm{NS}^{\mu}_{\kappa}$ case appears when $\kappa = \mu^{++}$. It became impossible anymore to use submodels which are μ-closed. For non μ-closed submodels N_0, N_1 (even if they are good), it is unclear, in general, whether starting at some $\mu_0 < \kappa$ all the indiscernibles $\delta \in N_0 \cap N_1$ for κ above μ_0 are for the same measures in N_0 and N_1. Recall that this property was used already in the definition of α^*. Also some new problems arise further in Lemmas 2.9, 2.11, 2.15 since ordinals of cofinality ω_1 will never be a club.

Let us first give the new definition of α^*. Consider in V the set $Z \subseteq \kappa$ consisting of ordinals δ so that

$$\text{(1)} \qquad \mathrm{cf}\,\delta = \omega_1 \quad \text{and}$$
$$\text{(2)} \qquad \mathrm{cf}^{\mathcal{K}(\mathcal{F})}\,\delta = \delta$$

are contained in a good for δ submodel N^*_δ so that there exists a sequence $\langle \delta_i \mid i < \omega \rangle \in N^*_\delta$ of indiscernibles for δ so that $\langle \beta^{N^*_\delta}(\delta_i) \mid i < \omega_1 \rangle \in N^*_\delta$ and for every indiscernible $\tau \geq \delta_0$ for δ in N^*_δ there is $i < \omega_1$ so that $\tau < \delta_i$, $\beta^{B^*_\delta}(\tau) < \beta^{N^*_\delta}(\delta_i)$, where $\beta^{N^*}_\delta(-)$ stand for the measure over δ itself.

LEMMA 2.17.1: *The set Z is stationary.*

Proof: Suppose otherwise. Let C be a club disjoint from Z. Let α be the least so that some condition in $(\mathrm{NS}^{\aleph_1}_{\aleph_3})$ forces that the measure $\mathcal{F}(\kappa,\alpha)$ is used to move κ in the generic ultrapower. Let M be a generic ultrapower defined by a generic elementary embedding using the measure $\mathcal{F}(\kappa,\alpha)$. Apply Lemma 2.1 to C inside M. Let $\langle \tau_i \mid i<\omega_1,\ i$ is a successor ordinal $\rangle$ be as in Lemma 2.1. Define $\tau_i=\bigcup_{j<i}\tau_j$ for a limit $i<\omega_1$. Let N be a submodel so that C, $\langle \tau_i \mid i<\omega_1\rangle \in N$, $|N|=\aleph_1$ and ${}^{\omega}N\subseteq N$. Suppose that $N \prec H_\chi$ for χ large enough so that for every $\delta\in N$ of cofinality ω_1 which was regular in $\mathcal{K}(\mathcal{F})$, N contains a counterexample N_δ to the condition (2). Pick a submodel $N^*\supseteq N$ of cardinality $\aleph_1$, closed under ω-sequences so that $N\cap\tau_i\in N^*$ for every i. For a successor $i<\omega_1$ denote $\bigcup(N\cap\tau_i)$ by δ_i. Then $\delta_i\in C$ and $\mathrm{cf}\,\delta_i=\omega_1$. Pick a club of indiscernibles of N^* for δ_i, $\langle\delta_i(j)\mid j<\omega_1\rangle$. Pick $N^{**}\supseteq N^*$ of cardinality $\aleph_1$, closed under ω-sequences and so that $h^{N^*}\in N^{**}$, $\big\langle\langle\delta_i(j)\mid j<\omega_1\rangle \bigm| i<\omega_1,$ i is a successor ordinal$\big\rangle\in N^{**}$, $\big\langle\langle\beta^{N^*}\big(\delta_i(j)\big)\mid j<\omega_1\rangle \bigm| i<\omega_1,\ i$ is a successor ordinal$\big\rangle\in N^{**}$, where $\beta^{N^*}\big(\delta_i(-)\big)$ stands for measures over δ_i. By [Mi4, Lemma 1.6], for every limit $i<\omega_1$ there exists $i'<i$ so that every indiscernible of N, μ, for κ, $\tau_{i'}<\mu<\tau_i$ is an indiscernible of N^{**} for κ for the same measure over τ_i. Using the Födor Lemma and the coherence function find $i_0<\omega_1$ so that for every limit $i>i_0$ every indiscernible μ of N^* for κ, $\tau_{i_0}<\mu<\tau_i$ is an indiscernible of N^{**} for κ for the same measure over τ_i. Once more using the coherence, this will imply that for every successor i large enough a final segment of $\langle\delta_i(j)\mid j<\omega_1\rangle$ will be indiscernibles for κ in N^{**} and over δ_i, $\beta^{N^*}\big(\delta_i(j)\big)=\beta^{N^{**}}\big(\delta_i(j)\big)$. Then N^{**} is a model satisfying the conditions (1) and (2) for δ_i. Contradiction. ∎

For $\xi<\kappa$ consider the set $Z_\xi=\big\{\delta\in Z\mid$ for every N_δ, $|N_\delta|=\aleph_0$, $N^*_\delta\supseteq N_\delta$ as in the definition of Z can be picked so that the parameter of N^*_δ is below ξ, i.e., $N^*_\delta\cap H_\chi\subseteq h^{N^*_\delta}(\xi\cup C^{N^*_\rho})\big\}$.

There exists $\xi_0<\kappa$ such that Z_{ξ_0} is stationary. Otherwise just pick for every $\xi<\kappa$ a club C_ξ disjoint to Z_ξ. Set $C=\triangle_{\xi<\kappa}C_\xi$. Use this C in the proof of Lemma 2.17.1 in order to obtain the contradiction.

Further, we shall deal with indiscernibles above this ξ_0, without stating it specifically.

Let us call further a submodel good for δ if it has the properties of N^*_δ in the definition of $Z_{\xi'_0}$. The proof of Mitchell's Lemma 1.6 from [Mi4] implies that, if $N_0 \subseteq N_1$ are good for δ, then the final segment of indiscernibles of N_0 for δ are indiscernibles of N_1 for δ for the same measures over δ.

Define now α^* as in the $\aleph_0$-case only replacing Y there on Z_{ξ_0}. Let us preserve the notations made there with obvious changes of ω with ω_1.

PROPOSITION 2.17.2: *If* $NS^{\aleph_1}_{\aleph_3}$ *is precipitous and* $2^{\aleph_0} = \aleph_1$ *then there exists an* $\aleph_2$*-repeat point.*

Let us split the proof into few claims.

CLAIM A: *For every submodel* N' *there exists* $N \supseteq N'$ *so that for every* $N'' \supseteq N$, *for every* $\alpha < \omega_1$, *for every infinite* $s \subseteq \omega$, *for every sequence* $\langle \mu'_n \mid n \in s\rangle$ *of indiscernibles of* N' *for* $\tau_{\alpha+\omega}$ *so that* $\mu'_n \in (\tau_{\alpha+n}, \tau_{\alpha+n+1})$, *the following holds:*

If there is a sequence $\langle \mu''_n \mid n \in s\rangle$ *of indiscernibles of* N'' *for* $\tau_{\alpha+\omega}$ *so that* $\mu''_n \in (\mu'_n, \tau_{\alpha+n+1})$ *and* $\beta^{N''}_\alpha(\mu''_n) > \beta^{N'}_\alpha(\alpha'_n)$ *for all but finitely many* n*'s in* s, *then there is a sequence* $\langle \mu_n \mid n \in s\rangle$ *of indiscernibles of* N *for* $\tau_{\alpha+\omega}$ *so that* $\mu_n \in (\mu'_n, \tau_{\alpha+n+1})$ *and* $\beta^{N}_\alpha(\mu_n) > \beta^{N'}_\alpha(\mu'_n)$ *for all but finitely many* n*'s in* s, *where* $\beta_\alpha(\mu)$ *denotes the measure for* μ *over* α.

Proof: Suppose otherwise. Let N' witness this. Let $\langle \vec{\mu}_\alpha \mid \alpha < \omega_1\rangle$ be an enumeration of all possible sequences of indiscernibles of N_0. Define by induction an increasing continuous sequence $\langle N_\alpha \mid \alpha < \omega_1\rangle$, so that $N_{\alpha+1}$ takes care on $\vec{\mu}_\alpha$ when it is possible. Set $N = \bigcup_{\alpha<\omega_1} N_\alpha$. There are $N'' \supseteq N$, $\alpha < \omega_1$, $s \subseteq \omega$ a sequence $\langle \mu'_n \mid n \in s\rangle$ of indiscernibles of N' for $\tau_{\alpha+\omega}$ so that $\mu'_n \in (\tau_{\alpha+n}, \tau_{\alpha+n+1})$, so that $\mu''_n > \mu'_n$ and $\beta^{N''}_\alpha(\mu''_n) > \beta^{N}_\alpha(\mu'_n)$ for all but finitely many n's in s but there is no sequence $\langle mu_n \mid n < \omega\rangle$ of N satisfying this. Let $\beta < \omega_1$ be so that $\vec{\mu}_\beta = \langle mu'_n \mid n \in s\rangle$. Then, by the definition of $N_{\beta+1}$, there exists a sequence $\langle \mu^\beta_n \mid n \in s\rangle$ of indiscernibles of $N_{\beta+1}$ for $\tau_{\alpha+\omega}$ so that $\beta^{N_{\beta+1}}_\alpha(\mu^\beta_n) > \beta^{N'}_\alpha(\mu'_n)$ and $\mu^\beta_n > \mu'_n$ for all but finitely many n's in s. But $\beta^{N}_\alpha(\mu^\beta_n) = \beta^{N_{\beta+1}}_\alpha(\mu^\beta_n)$ for all but finitely many n's in s. Contradiction. ∎

CLAIM B: *For every* N' *there exists* $N \supseteq N'$ *so that* ${}^\omega N \subseteq N$ *and for every* $N'' \supseteq N$, *for every* $\alpha < \omega$, $s \subseteq \omega$ *and sequence* $\langle \mu'_n \mid n \in s\rangle$ *of indiscernibles of* N *for* $\tau_{\alpha+\omega}$ *so that* $\mu'_n \in (\tau_{\alpha+n}, \tau_{\alpha+n+1})$, *the following holds:*

If there is a sequence $\langle \mu''_n \mid n \in s\rangle$ *of indiscernibles of* N'' *for* $\tau_{\alpha+\omega}$ *so that* $\mu''_n \in (\mu'_n, \tau_{\alpha+1})$ *and* $\beta^{N''}_\alpha(\mu''_n) > \beta^{N}_\alpha(\mu'_n)$ *for all but finitely many* n*'s, then there exists such a sequence already in* N.

Proof: Use Claim A ω_1-times. ∎

Let $\mu \in C$ be an indiscernible for $\tau_{\alpha+\omega}(\alpha < \omega_1)$ in some N. Denote by $\beta^{*\alpha}(\mu)^N$ the maximal $\beta < \beta^N(\mu)$ such β. Define $\langle(\beta_n^{*\alpha})^N \mid n < \omega\rangle$, $\langle(d_n^\alpha)^N \mid n < \omega\rangle$ as in the $\aleph_0$-case but use $\tau_{\alpha+\omega}$ instead of κ.

CLAIM C: *For every N' there exists $N \supseteq N'$ so that:*

(a) *for every $\alpha < \omega_1$ $\langle(\beta_n^{*\alpha})^N \mid n < \omega\rangle$ exists,*

(b) *for every $N'' \supseteq N'$, for every $\alpha < \omega_1$ if $\langle(\beta_n^{*\alpha})^{N''} \mid n < \omega\rangle$ exists, then $(\beta_n^{*\alpha})^{N''} \leq (\beta_n^{*\alpha})^N$ for all but finitely many n's.*

Proof: Suppose otherwise. Let N_0 witness this. Split ω_1 into ω_1-disjoint sets $\langle s_{i+1} \mid i < \omega_1\rangle$ of cardinality ω_1. Define by induction an increasing continuous sequence $\langle N_i \mid i < \omega_1 \cdot \omega_1\rangle$.

Suppose $\langle N_{i'} \mid i' \leq i\rangle$ is defined. Define N_{i+1}. Let $N_i' \supseteq N_i$ be ω-closed and so that $\langle N_i \cap \tau_\alpha \mid \alpha < \omega_1\rangle \in N_i'$. Pick $N_i'' \supseteq N_i'$ to be as in Claim B. Let $i = \omega_1 \cdot i_0 + i_1$, where $i_1 < \omega_1$. Pick $\alpha < \omega_1$ so that $i_1 \in S_\alpha$. If $\langle(\beta_n^{*\alpha})^{N_i''} \mid n < \omega\rangle$ are undefined and there exists $N \supseteq N_i''$ with $\langle(\beta_n^{*\alpha})^N \mid n < \omega\rangle$ defined, then let N_{i+1} be some such N. If $\langle(\beta_n^{*\alpha})^{N_i''} \mid n < \omega\rangle$ are undefined and there is no N as above, then let $N_{i+1} = N_i'$. If $\langle(\beta_n^{*\alpha})^{N_i''} \mid n < \omega\rangle$ are defined and there is no $N \supseteq N_i''$ with $\langle(\beta_n^{*\alpha})^N \mid n < \omega\rangle$ defined and so that $(\beta_n^{*\alpha})^N > (\beta_n^{*\alpha})^{N_i''}$ for infinitely many n's, then let N_{i+1} be some such N. Otherwise, let $N_{i+1} = N_i''$. It completes the definition of $\langle N_i \mid i < \omega_1 \cdot \omega_1\rangle$. Set $N = \bigcup\{N_i \mid i < \omega_1 \cdot \omega_1\}$. Let $\alpha < \omega_1$. We like to show that $(\beta_n^{*\alpha})^N$'s exist. Let us first show the following:

SUBCLAIM: For every $\xi < \omega_1 \cdot \omega_1$ of cofinality ω_1, for all but finitely many n's the following holds:

For every indiscernible ρ such that $\tau_{\alpha+n+1} > \rho \geq \min(d_n^\alpha)^{N_\xi}$ of N_ξ for $\tau_{\alpha+\omega}$ there exists an indiscernible $\delta \in C$, $\tau_{\alpha+n+1} > \delta > \rho$ of N_ξ for $\tau_{\alpha+\omega}$ of cofinality ω_1 so that $\beta_\alpha^{N_\xi}(\delta) > \beta_\alpha^{N_\xi}(\rho)$.

Proof: Suppose otherwise. Let $\langle\rho_n \mid n \in s\rangle$ witness this. Deal for simplicity with $\xi = \omega_1$. Find $i \in S_\alpha$ so that $\{\rho_n \mid n \in s\} \subseteq N_i$. By the definition of N_{i+1} it contains N_i''. Then for almost all n's in s, $\xi_n = \bigcup N_i'' \cap \tau_{\alpha+n+1}$ is in C is of cofinality ω_1 and it is a limit of indiscernibles ξ of N_i'' satisfying $\beta_\alpha^{N_i''}(\xi) > \beta_\alpha^{N_{\omega_1}}(\rho_n)$. By [Mi4, Lemma 1.6], then $\beta_\alpha^{N_{\omega_1}}(\xi_n) > \beta_\alpha^{N_{\omega_1}}(\rho_n)$ for all but finitely many n's in s. Contradiction. ∎

So, for all but finitely many n's, $\beta_\alpha\big(\bigcup(N_\xi \cap \tau_{\alpha+n})\big)^{N_{\xi+1}}$ exists. Denote $\bigcup(N_\xi \cap \tau_{\alpha+n})$ by $\delta_n^{\alpha,\xi}$.

The proof similar to those of the subclaim gives that for all but finitely many n's, the following holds:

For every indiscernible ρ, $\tau_{\alpha+n+1} > \rho \geq \min(d_n^\alpha)^N$ of N for $\tau_{\alpha+\omega}$ $\beta_\alpha^N(\delta_n^{\alpha,\xi}) > \beta_\alpha^N(\rho)$ for all but countably many ξ's $\xi < \omega_1 \cdot \omega_1$ and $\operatorname{cf}\xi = \omega_1$.

It implies that, starting with some ξ_0, all $\beta_n^{*\alpha}(\delta_n^{\alpha,\xi})^N$'s should be the same. Since otherwise, just consider some $N^* \supseteq N \cup \{h^N\} \cup \{\langle \delta_n^{\alpha,\xi}, \beta_n^\alpha(\delta_n^{\alpha,\xi})\rangle \mid n < \omega\ ,\ \xi < \omega_1 \cdot \omega_1\} \cup \{\langle \bigcup N \cap \tau_{\alpha+\omega}\rangle \mid n < \omega\}$ and apply the analog of Lemma 2.9 to obtain the contradiction. So $(\beta_n^{*\alpha})^N$ exists for all but finitely many n's.

Since the sequence $\langle N_i \mid i < \omega_1 \cdot \omega_1\rangle$ is increasing and continuous the same is true for a club many N_i's and with $(\beta_n^{*\alpha})^{N_i} = (\beta_n^{*\alpha})^N$. Now, as in Lemma 2.10.2, we obtain a contradiction, since $\neg(b)$ was used a lot of times in the definition of $\langle N_i \mid i < \omega_1 \cdot \omega_1\rangle$. ∎

Using Claim C, it is not hard to prove the following analog of Lemma 2.10.2.

Claim D: *There exists a sequence $\langle \beta_n^{*\alpha} \mid n < \omega\ ,\ \alpha < \omega_1\rangle$ so that for every N' there is N as in Claim C, so that for every $\alpha < \omega_1$, $\beta_n^{*\alpha} = (\beta_n^{*\alpha})^N$ holds for all but finitely many n's.*

Claim E: *For every $\delta < \aleph_2$, for every N' there exists $N \supseteq N'$ so that for every $\alpha < \omega_1$, $(\beta_n^{*\alpha})^N = \beta_n^{*\alpha}$ and $\delta_n^{\alpha N} \geq \beta_n^{*\alpha} + \delta$ holds for all but finitely many n's, where $\delta_n^{\alpha N} = \sup\{\beta_\alpha^N(\tau) \mid \tau \in C \cap \tau_{\alpha+n+1} - \min d_n^{\alpha N}\}$.*

The proof is similar to Lemma 2.11 and we leave it to the reader.

The proof of the proposition follows now from Claim E as in the $\aleph_0$-case. Notice only, that all but countably many τ_α's remain indiscernibles for κ in any N. So $A \cap \tau_{\alpha+\omega} \in \mathcal{F}(\tau_{\alpha+\omega}, \beta_n^{*\alpha} + \delta)$ for ω_1 α's, will imply that $A \in \mathcal{F}(\kappa, \beta)$ for some $\beta = \beta^* + \delta$, where β^* is the maximal ordinal below β such that $A^* \in \mathcal{F}(\kappa, \beta^*)$.

The proof that $\operatorname{cf}\alpha^* = \omega$ generalizes straightforwardly to the ω_1-case, just in Lemma 2.16, involves $\beta^\alpha(-)^N, \beta^{*\alpha}(-)^N$ for every $\alpha < \omega_1$ and use Claim B in the construction of the sequence of models there. ∎

Theorem 2.18.0: *Suppose that $\mathrm{NS}_\kappa^{\aleph_0}$ is precipitous and the empty condition forces "$\check{\kappa} \geq (2^\omega)^+$". Then there exists an up-repeat point.*

Let us assume now that $\mathrm{NS}_\kappa^{\aleph_0}$ is precipitous and in generic extensions by $\mathrm{NS}_\kappa^{\aleph_0}$ always $\kappa \geq (2^\omega)^+$. Then it is possible to strengthen Lemmas 2.1 and 2.2 as follows.

LEMMA 2.18: *There exists a sequence $\langle \tau_n \mid n < \omega\rangle$ such that for every club $C \in V$ a final segment of $\langle \tau_n \mid n < \omega\rangle$ satisfies the conditions (i), (ii), (iv) of Lemmas 2.1 or 2.2.*

Proof: Suppose otherwise. Let $\langle C_\nu \mid \nu < (\kappa^+)^V\rangle$ be an enumeration of a generating family of clubs in V so that $C_\nu - C_{\nu'}$ is nonstationary for every $\nu \geq \nu'$. In a generic extension $V[G]$ of V by $\mathrm{NS}_\kappa^{\aleph_0}$ construct an elementary chain $\langle N_\mu \mid \nu < \left((2^\omega)^+\right)^{V[G]}\rangle$ so that

(1) $N_\nu \supseteq \bigcup_{\nu'<\nu}(N_{\nu'} \cup \{h^{N_{\nu'}}\})$,

(2) $|N_\nu| \leq (2^\omega)^{V[G]}$,

(3) ${}^\omega N_\nu \subseteq N_\nu$,

(4) for every $\nu > \nu'$ there exists $n(\nu', \nu) < \omega$ s.t. (i) for every $n \geq n(\nu', \nu)$ $\tau_n^{N_\nu} \leq \tau_n^{N_{\nu'}}$ and (ii) for infinitely many n's $\tau_n^{N_\nu} < \tau_n^{N_{\nu'}}$.

Start with any N_0 s.t. $C_0 \in N_0$ and satisfying (2), (3). By the assumption there exists ν_1 and N' s.t. $C_{\nu_1} \in N'$ and $\langle \tau_n^{N_0} \mid n < \omega\rangle$, $\langle \tau_n^{N'} \mid n < \omega\rangle$ satisfy (4). Pick $\tau_n^{N'}$ as large below $\tau_n^{N_0}$ as possible.

Pick N_1 to be some N s.t. $N \supseteq N_0 \cup N' \cup \{h^{N_0}, h^{N_1}\}$, ${}^\omega N \subseteq N$, $|N| \leq 2^\omega$. Suppose that $\langle N_\nu \mid \nu < \bar\nu\rangle$ and a subsequence $\langle C_{i_\nu} \mid \nu < \bar\nu\rangle$ of $\langle C_i \mid i < (\kappa^+)^V\rangle$ are constructed, $C_{i_\nu} \in N_\nu$. If $\bar\nu$ is a successor ordinal then define $N_{\bar\nu}$ as above. Assume that $\bar\nu$ is a limit ordinal.

Since κ changes its cofinality to ω still remaining $\geq \left((2^\omega)^+\right)^{V[G]}$, the cofinality of $(\kappa^+)^V$ should be $\geq (2^\omega)^+$ in the generic extension. So the sequence $\langle i_\nu \mid \nu < \bar\nu\rangle$ is bounded in $(\kappa^+)^V$. Let $i_{\bar\nu} = \min\left((\kappa^+)^V - \bigcup_{\nu<\bar\nu} i_\nu\right)$. Pick $N_{\bar\nu}$ to be any N satisfying (1)–(3) and containing $C_{i_{\bar\nu}}$. Then $C_{i_{\bar\nu}}$ is almost contained in every C_{i_ν} $(\nu < \bar\nu)$. So (4) is satisfied since $\bar\nu$ is a limit ordinal. It completes the construction of $\langle N_\nu \mid \nu < (2^\omega)^+\rangle$. Define now a partition $f\colon [2^{\omega^+}]^2 \to \omega$ $f(\nu', \nu) = \min\{n \mid \tau_n^{N_{\nu'}} > \tau_n^{N_\nu}\}$. By Erdös–Rado, there exists a homogeneous set of cardinality 2^ω for f. But it is impossible since there is no infinite decreasing sequences of ordinal. Contradiction. ∎

We would like to show that there exists an up-repeat point. Suppose otherwise. Pick a sequence $\langle \tau_n \mid n < \omega\rangle$ as in Lemma 2.18. Let α^* be as defined above. Let us deal with α^* itself.

LEMMA 2.18.1: *For all but finitely many n's* $\mathrm{cf}\,\tau_n > \aleph_0$.

Proof: Suppose otherwise. Fix some N with $\langle \tau_n \mid n < \omega\rangle \in N$. Suppose for simplicity that each τ_n has cofinality ω. Then $(\beta^*(\tau_n))^N$ is defined for every n.

Now pick some $D \in \mathcal{F}(\kappa, \alpha^*)$ s.t. $D \notin \mathcal{F}(\kappa, (\beta^*(\tau_n))^N)$. Replacing now A^* by $D \cap A^*$, we find a club C' which avoids τ_n's. This means that $\langle \tau_n \mid n < \omega \rangle$ can be replaced by a smaller sequence, which contradicts the choice $\langle \tau_n \mid n < \omega \rangle$. ∎

Remark 2.18.2: Using a similar argument it is possible to show that $\langle \tau_n \mid n < \omega \rangle$ should satisfy the condition (iii) of Lemma 2.1 or 2.2.

Consider now a few cases. We preserve the notation used above.

CASE 1: cf $\alpha^* = \kappa^+$. Define a chain $\langle N_\nu \mid \nu < \omega_1 \rangle$ of submodels in M so that

(i) $N_\nu \supseteq \bigcup_{\nu' < \nu} (N_{\nu'} \cup \{h^{N_{\nu'}}\})$,

(ii) $\beta_n^{*N_\nu} \notin \bigcup \{\beta_n^{*N_{\nu'}} \mid \nu' < \nu\}$ for all but finitely many n's.

In order to insure (ii) notice that there exists a set $A_\nu \in \mathcal{F}(\kappa, \alpha^*)$ such that $A_\nu \subseteq A^*$ and $A_\nu \notin \bigcup \{\mathcal{F}(\kappa, \beta_n^{*N_{\nu'}}) \mid n < \omega,\ \nu' < \nu\}$. Replace A^* by A_ν and define N_ν according to A_ν.

Finally, set $N = \bigcup_{\nu < \omega_1} N_\nu$. Then for some $n_0 < \omega$ there exists an increasing sequence of indiscernibles $\{\mu_m \mid m < \omega\} \subseteq (\tau_{n_0}, \tau_{n_0+1})$ with $\langle \beta^{*N}(\mu_m) \mid m < \omega \rangle$ also increasing. But it is n_0 impossible by Lemma 2.9.

CASE 2: cf $\alpha^* = \kappa$. Let $\langle C_\nu^\square \mid \nu < \kappa^{++} \rangle$ be the coherent box sequence over κ in $\mathcal{K}(\vec{\mathcal{F}})$, i.e., $\square_{\kappa^+}$-sequence such that $\square_{\kappa^+}$-sequences of ultrapowers by measures over κ are initial segments of it. We refer to Schimmerling [Sc] for the fact of existence of such sequences. For $\nu < \kappa^{++}$, $\delta < \kappa^+$ let $C_\nu^\square(\delta)$ be the δ'th element of $C_\nu^\square$ if there exists such an element.

Pick, in $\mathcal{K}(\vec{\mathcal{F}})$, sets $\langle A(\delta) \mid \delta < \kappa \rangle$ such that $A(\delta) \in \mathcal{F}(\kappa, C_{\alpha^*}^\square(\delta))$ and $A(\delta)$ does not belong to any $\mathcal{F}(\kappa, \beta)$ with $\beta > C_{\alpha^*}^\square(\delta)$. It is possible since we assumed that there is no up-repeat points.

We'll use the sequence $\langle A(\delta) \mid \delta < \kappa \rangle$ instead of A^*.

Let N be a submodel. Suppose that $\langle \mu_n \mid n < \omega \rangle$ is a sequence of indiscernibles for κ s.t. $\mu_n \in (\min d_n^N, \tau_n)$, where d_n^N is as in Lemma 2.8. Let $n < \omega$. Set $m(n)$ to be the least m so that $C_{\alpha^*}^\square(\tau_m) \neq C_{\beta^*(\mu_n)}^\square(\tau_m)$. Pick a set $A_n \in \mathcal{F}(\kappa, C_{\alpha^*}^\square(\tau_{m(n)})) - \mathcal{F}(\kappa, C_{\beta^*(\mu_n)}^\square(\tau_{m(n)}))$, $A_n \subseteq A(\tau_{m(n)})$. Let $A_n^* \in \mathcal{F}(\kappa, \alpha^*)$ be a set of all $\delta, \tau_{m(n)} < \delta < \kappa$ s.t. $A_n \cap \delta \in \mathcal{F}(\delta, C_{O^{\mathcal{F}}(\delta)}^\square(\tau_{m(n)}))$. Since cf $\kappa^+ > \omega$, there exists a set $A^* \in \mathcal{F}(\kappa, \alpha^*)$ which is almost contained in every A_n^* and does not belong to any measure above α^*. Define $B(A^*)$ in $\mathcal{K}(\mathcal{F})$ to be the set of all $\delta < \kappa$ such that there exists largest $\delta^* < \delta$ s.t. $A^* \cap \delta \in \mathcal{F}(\delta, \delta^*)$ and cf $\delta^* = \kappa$. Then $A^* \cup B(A^*)$ belongs to all relevant measures and hence contains an ω-club.

Define now a chain $\langle N_\nu \mid \mu < \omega_1\rangle$ and sequences $\langle \mu_{n\nu} \mid n < \omega, \nu < \omega_1\rangle$, $\langle A_\nu \mid \nu < \omega_1\rangle$, $\langle C_\nu \mid \nu < \omega_1\rangle$ so that

(0) $\langle A(\delta) \mid \delta < \kappa\rangle \in N_0$,

(1) $N_\nu \supseteq \bigcup_{\nu'<\nu}(N_{\nu'} \cup \{h^{N_{\nu'}}\}$,

(2) for $n < \omega$, if τ_n is an indiscernible for κ in N_ν then $\langle \mu_{n\nu} \mid n < \omega\rangle$ is a sequence of indiscernibles for κ in N_ν s.t. $\mu_{n\nu} \in (\min d_n^{N_\nu}, \tau_n)$ otherwise $\mu_{n\nu} = 0$,

(3) $A_\nu = (A^*)^{N_\nu}$, i.e., defined as above using N_ν and $\langle \mu_{n\nu} \mid n < \omega\rangle$,

(4) $C_\nu \subseteq A_\nu \cup B(A_\nu)$ is a club in V,

(5) $A_\nu, C_\nu \in N_{\nu+1}$,

(6) A_ν is almost contained in $A_{\nu'}$ and C_ν in $C_{\nu'}$ for $\nu' < \nu$,

(7) $\langle d_n^{N_{\nu+1}} \mid n < \omega\rangle$ is defined as in Lemma 2.8 using C_ν,

(8) $\mu_{n\nu+1} > \mu_{n\nu'}$ for every $n < \omega$, $\nu' \le \nu$,

(9) $N_{\nu+1}$ satisfies conditions (a)–(e) from page 77 with C replaced by C_ν, A by A_ν.

Set $N = \bigcup_{\nu<\omega_1} N_\nu$. Find $S \subseteq \omega_1$ of cardinality ω_1 consisting of successor ordinals and $n_0 < \omega$ such that for every $\nu \in S$, every $n \ge n_0$, the following conditions hold:

(a) τ_n is an indiscernible for κ in N_ν and in N,

(b) $\mu_{n\nu}$ is an indiscernible for κ in N and $\beta^N(\mu_{n\nu}) = \beta^{N_\nu}(\mu_{n\nu})$,

(c) the support of $A_{\nu-1}$ in N and N_ν is below τ_{n_0}.

Pick an increasing sequence $\nu_1 < \nu_2 < \cdots < \nu_k < \cdots$ of elements of S and some $n \ge n_0$ such that $\min d_n^N < \mu_{n\nu_1} < \mu_{n\nu_2} < \cdots < \mu_{n\nu_k} < \cdots$. Let us denote $\mu_{n\nu_k}$ simply by μ_k. $\beta^N(\mu_1) < \beta^N(\mu_2) < \cdots < \beta^N(\mu_k) < \cdots$ by the definition of d_n^N. Set $\mu = \bigcup_{k<\omega} \mu_k$. Then without loss of generality $\beta^N(\mu) < \alpha^*$, since otherwise just replace n by some $n' > n$. Recall that by the choice of α^* it is impossible to have an unbounded in κ sequence of indiscernibles μ with $\beta^N(\mu) \ge \alpha^*$.

By the choice of μ_k $(k < \omega)$ and (c), $A_{\nu_k-1} \in \mathcal{F}(\kappa, \beta^{*N_{\nu_k}}(\mu_k))$. Let $i < j < \omega$. Since by (6) $A_{\nu_{j-1}}$ is almost contained in A_{ν_i}, $A_{\nu_i} \in \mathcal{F}(\kappa, \beta^{*N}(\mu_j))$. But $A_{\nu_i} = (A^*)^{N_{\nu_i}}$. Hence $(A_n^*)^{N_{\nu_i}}$ is almost contained in A_{ν_i}. So $(A_n^*)^{N_{\nu_i}} \in \mathcal{F}(\kappa, \beta^{*N}(\mu_j))$. Then $A_n^{N_{\nu_i}} \in \mathcal{F}(\kappa, C^{\square}_{\beta^{*N}(\mu_j)}(\tau_{m(n)}))$. But $A_n^{N_{\nu_i}} \notin \mathcal{F}(\kappa, C^{\square}_{\beta^{*N}(\mu_i)})$ $(\tau_{m(n)})$. Hence $C^{\square}_{\beta^{*N}(\mu_j)}(\tau_{m(n)}) \ge \beta^N(\mu_i)$, since between $C^{\square}_{\beta^{*N}(\mu_i)}(\tau_{m(n)})$ and $\beta^N(\mu_i)$ there is no measure to which $A(\tau_{m(n)})$ belongs. Pick the least $\bar{n} < \omega$ s.t. $C_{\beta^{*N}(\mu)}(\tau_{\bar{n}}) < C_{\alpha^*}(\tau_{\bar{n}})$. Then for every $k < \omega$ $(\tau_{m(n)})^{N_{\nu_k}} < \tau_{\bar{n}}$, so for every $i <$

$j < \omega$, $\beta^N(\mu_i) < C^{\square}_{\beta^{*N}(\mu_j)}(\tau_{\bar n})$. But then $\bigcup_{i<\omega} \beta^N(\mu_i)$ is bounded below $\beta^{*N}(\mu)$, since otherwise $A(\tau_{\bar n})$ would belong to unboundedly many in $\beta^{*N}(\mu)$ measures, which contradicts the choice of $A(\tau_{\bar n})$. This is impossible by (9). Contradiction.

CASE 3: *cf $\alpha^* < \kappa$ or α^* is a successor ordinal.* Similar to Cases 2 and 3.

This completes the proof of 2.18.0. ∎

Lemma 2.18 can be used to show the following

THEOREM 2.19: *Suppose that NS_κ is κ^+-saturated for $\kappa \geq (2^\omega)^+$. Then $\exists\kappa$ $O(\kappa) = \kappa^{++}$ in an inner model.*

Proof: Suppose that $\neg\exists\kappa\ O(\kappa) = \kappa^{++}$. By Shelah [S], κ is weakly inaccessible. So $\kappa > (2^\omega)^+$. Then the empty condition forces "$\kappa \geq (2^\omega)^+$". Hence the condition Lemma 2.18 is satisfied if κ changes its cofinality to ω in a generic ultrapower.

CLAIM 2.19.1: *κ changes cofinality to ω in a generic ultrapower using the least relevant measure.*

Proof: Suppose otherwise. Let M be a generic ultrapower using the least relevant measure. κ^+-saturatedness implies that the total number of relevant measures is $\leq \kappa$.

Pick a sequence $\langle A_\nu \mid \nu < \kappa\rangle$ of disjoint sets s.t. A_ν belongs to ν'th relevant measure and $\min A_\nu > \nu$. Then $A = \bigcup_{\nu<\kappa} A_\nu$ contains a club and at least one of the A_ν's remains stationary in M. Let A_{ν_0} be a stationary in M. But then ν_0-th relevant measure can be defined in M as the filter dual to $NS_\kappa | A_{\nu_0} = \{B \subseteq \kappa \mid B \cap A_{\nu_0}$ is nonstationary$\}$, which is impossible. Contradiction. ∎

Let M be a generic ultrapower using the least relevant measure. By the claim and Lemma 2.18 find the sequence $\langle \tau_n \mid n < \omega\rangle$. Let N be a submodel of H_{λ^+} (in M) with $\langle \tau_n \mid n < \omega\rangle \in N$. By removing an initial segment of τ_n's assume that all τ_n's are indiscernibles for $\bar\kappa^N$. Assume also simplification of the notions, that $\bar\kappa^N = \kappa$.

Consider the set $\{\beta^N(\tau_n) \mid n < \omega\}$. By κ^+-c.c., find a set A in $\mathcal{K}(\vec{\mathcal{F}})$ of cardinality $\leq \kappa$ containing $\{\beta^N(\tau_n) \mid n < \omega\}$. Pick A such that $\bigcup A \leq$ the least relevant ordinal. Find a set B in the intersection of all relevant measures which does not belong to $\bigcup\{\mathcal{F}(\kappa, \alpha) \mid \alpha \in A\}$. Then B contains a club C in V. Pick now $N_1 \supseteq N \cup \{h^N\} \bigcup\{B, C\}$. The final setment of $\langle \tau_n \mid n < \omega\rangle$ does not belong to B and so also to C. But C is a club. Hence there is $n_0 < \omega$ such that for every

$n > n_0$, C is bounded in τ_n. But it contradicts Lemma 2.18. Contradiction. ∎

3. Constructions of precipitous ideals

Denote by $\mathrm{NS}_\kappa^{\mathrm{Sing}}$ the ideal $\mathrm{NS}_\kappa \restriction \{\alpha < \kappa \mid \alpha \text{ is singular and } (\mathrm{cf}\ \alpha)^+ < \kappa\}$. Our aim will be to prove the following.

THEOREM 3.1:

(1) *Suppose that there exists an $(\omega, \kappa^+ + 1)$-repeat point at κ in $\mathcal{K}(\vec{\mathcal{F}})$. Then in a cardinal preserving generic extension "GCH + NS_κ is precipitous" holds.*

(2) *Suppose that there exists an (ω, κ^+)-repeat point at κ in $\mathcal{K}(\vec{\mathcal{F}})$. Then in a cardinal preserving generic extension GCH + $\mathrm{NS}_\kappa^{\mathrm{Sing}}$ is precipitous holds.*

THEOREM 3.2:

(1) *Suppose that for a regular cardinal $\lambda < \kappa$ there exists an $(\omega, \lambda + 1)$-repeat point in $\mathcal{K}(\vec{\mathcal{F}})$. Then in a generic extension, preserving all the cardinals $\leq \lambda$, "$\kappa = \lambda^+$, NS_κ is precipitous and GCH" holds.*

(2) *Suppose that for some regular $\lambda < \kappa$ there exists an (ω, λ)-repeat point at κ in $\mathcal{K}(\vec{\mathcal{F}})$. Then in a generic extension, preserving all the cardinals $\leq \lambda$, "$\kappa = \lambda^+$, $\mathrm{NS}_\kappa^{\mathrm{Sing}}$ is precipitous and GCH" hold.*

Suppose first that α is an $(\omega, \kappa^+ + 1)$-repeat point at κ in $\mathcal{K}(\vec{\mathcal{F}}) = V$. Force first with the forcing $\mathcal{P}_\kappa$ defined in [G2-3]. This forcing adds Prikry, Magidor or Radin sequences to every $\delta < \kappa$ with $O^{\vec{\mathcal{F}}}(\delta) > 0$ and it satisfies κ-c.c. Let Q_κ be the Backward–Easton iteration over $V^{\mathcal{P}_\kappa}$ which adds δ^+ Cohen subsets to every regular $\delta < \kappa$ with $O^{\vec{\mathcal{F}}}(\delta) > 0$. Fix a generic subset $G_\kappa * H_\kappa$ of $\mathcal{P}_\kappa * Q_\kappa$. The filter $F = \bigcap\{\mathcal{F}(\kappa, \beta) \mid \alpha \leq \beta \leq \alpha + \kappa^+\}$ is still precipitous in $V[G_\kappa * H_\kappa]$ since, by Magidor [Ma1], κ-c.c. extensions preserve precipitousness. We shall shoot clubs through every set in F, then through the sets of generic points and so on.

In order to preserve precipitousness we shall show that each elementary embedding by $\mathcal{F}(\kappa, \beta)$ can be extended at each stage.

The situation here is similar to those of [G1-2]. Let us concentrate only on the additional arguments needed in order to apply the forcing of [G1].

Let $j_\beta\colon V \longrightarrow N_\beta \simeq V^\kappa/\mathcal{F}(\kappa, \beta)$ be the canonical elementary embedding, where $\beta < O^{\vec{\mathcal{F}}}(\kappa)$.

LEMMA 3.3: *Let $\beta < O^{\vec{\mathcal{F}}}(\kappa)$ and $\mathcal{P}(\kappa,\beta)$ is the forcing used over κ in $j_\beta(\mathcal{P}_\kappa)$. Then $\Vdash_{Q_\kappa \times \mathcal{P}(\kappa,\beta)}$ "$\{\langle q_i \mid i < \mathrm{cf}\ \kappa\rangle \mid$ for every $j < \mathrm{cf}\ \kappa$ $\langle q_i \mid i < j\rangle \in \underset{\sim}{H_\kappa}\}$ is a $N_\beta^{\mathcal{P}_\kappa * \mathcal{P}(\kappa,\beta)}$-generic subset of Q_κ", where $\underset{\sim}{H_\kappa}$ stands for the canonical generic subset of Q_κ over $V^{\mathcal{P}_\kappa}$.*

Proof: If the forcing $\mathcal{P}(\kappa,\beta)$ preserves regularity of κ then the lemma is trivial. Suppose that $\Vdash_{\mathcal{P}(\kappa,\beta)}$ "cf $\kappa < \kappa$" and that the statement of the lemma is false.

Let $\langle q,p\rangle \in Q_\kappa \times \mathcal{P}(\kappa,\beta)$ forces the negation. Pick a generic subset G of $\mathcal{P}(\kappa,\beta)$ with $p \in G$. Find also a generic subset H of Q_κ, for Q_κ, as it is defined in $V[G_\kappa, G]$. Let $q \in H$. Consider the set $\bar{H} = \{s|i \mid s \in H,\ i < \kappa\}$. Clearly $q \in \bar{H}$. It is sufficient to show that $\bar{H}$ is a generic subset of Q_κ in the sense of $V[G_\kappa]$. Let S be a maximal antichain of $Q_\kappa^{V[G_\kappa]}$. Since κ is a measurable in $V[G_\kappa]$, for some regular $i < \kappa$, $S \subseteq Q_i$. Hence $S \cap H \neq \emptyset$. Then also $S \cap \bar{H} \neq 0$ since $S \cap H \subseteq S \cap \bar{H}$. ∎

Let $E \in \bigcap\{\mathcal{F}(\kappa,\beta) \mid \alpha < \beta \leq \alpha+\kappa^+\}$. Define, in $V[G_\kappa, H_\kappa]$, $P[E]$ to be the forcing notion consisting of all $d \subseteq E$, $|d| < \kappa$ and d is closed. For $d_1, d_2 \in P[E]$ set $d_1 \leq d_2$ (d_2 is stronger) if d_2 is an end extension of d_1.

α is a $\kappa^+ + 1$-repeat point, hence there is $\beta < \alpha$ s.t. $\beta + \kappa^+ + \kappa^+ < \alpha$ and $E \in \bigcap\{\mathcal{F}(\kappa,\beta') \mid \beta \leq \beta' \leq \beta + \kappa^+\}$. Then the set $E(\kappa^+) = \{\delta \in E \mid$ there is $\bar{\delta}$ s.t. $O^{\vec{\mathcal{F}}}(\delta) = \bar{\delta} + \kappa^+$ and $\delta \cap E \in \bigcap\{\mathcal{F}(\delta,\delta') \mid \bar{\delta} \leq \delta' < \bar{\delta} + \kappa^+\}\}$ belongs to $\mathcal{F}(\kappa, \beta + \kappa^+)$.

LEMMA 3.4: *For every $\delta \in E(\kappa^+)$ there exists a $V[G_\delta, H_\delta]$-generic subset of $P[E \cap \delta]$ inside $V[G_{\delta+1}, H_{\delta+1}]$.*

Proof: The forcing $\mathcal{P}_{\delta+1}/\mathcal{P}_\delta$ shoots a Radin club through $E \cap \delta$ without adding new bounded subsets of δ (see [G2-3]). Hence $E \cap \delta$ contains a club in $V[G_{\delta+1}, H_\delta]$ and the forcing $P[E \cap \delta]$ is the same over $V[G_\delta, H_\delta]$ and $V[G_{\delta+1}, H_\delta]$. But then $P[E \cap \delta]$ is isomorphic to the forcing for adding a Cohen subset of δ. So there exists even a $V[G_{\delta+1}, H_\delta]$ generic subset of $P[E \cap \delta]$ inside $V[G_{\delta+1}, H_{\delta+1}]$. ∎

Let $G_{\kappa+1} \times H_{\kappa+1}$ be a N_β-generic subset of $\mathcal{P}_{\kappa+1} \times Q_{\kappa+1}$, for some β, $\alpha < \beta < \alpha + \kappa^+ + 1$. In view of Lemma 3.3, let us not distinguish between H_κ and $H_{\kappa+1}|\kappa$.

LEMMA 3.5: *There exists a $V[G_\kappa, H_\kappa]$-generic subset of $P[E]$ inside $N_\beta[G_{\kappa+1}, H_{\kappa+1}]$.*

Proof: The proof is by induction on β. Let us consider only the first step $\beta = \alpha + 1$. The rest is as in Lemma 4.1 of [G2].

Note that cf $\kappa = \omega$ in $N_{\alpha+1}[G_{\kappa+1}, H_\kappa]$. Let $\langle \gamma_n \mid n < \omega \rangle$ be increasing and unbounded in α s.t. $E(\kappa^+) \in \mathcal{F}(\kappa, \gamma_n)$. It exists since α is an $(\omega, \kappa^+ + 1)$-repeat point. Pick an increasing unbounded in κ subsequence $\langle \delta_n \mid n < \omega \rangle$ of the generic sequence such that δ_n corresponds to the measure $\mathcal{F}(\kappa, \gamma_n)$. Without loss of generality assume that all δ_n's are in $E(\kappa^+)$. Using Lemma 3.4 construct a sequence of closed sets $\langle C_n \mid n < \omega \rangle$ so that

(1) $C_n \in V[G_{\delta_n+1}, H_{\delta_n+1}]$ is a $V[G_{\delta_n}, H_{\delta_n}]$-generic club through $E \cap \delta_n$,
(2) C_{n+1} is an endextension of $C_n \cup \{\delta_n\}$.

Set $C = \bigcup_{n<\omega} C_n$. Let us show that C is a $N_\alpha[G_\kappa, H_\kappa]$-generic club through E. Let $D \in V[G_\kappa, H_\kappa]$ be a dense open subset of $P[E]$. The set of ξ's such that $D \cap P[E \cap \xi] \in V[G_\xi, H_\xi]$ and $D \cap P[E \cap \xi]$ is a dense subset of $P[E \cap \xi]$ contains a final segment of δ_n's. But then C is an endextension of an element of D. ∎

Let E_1 be another set in $\bigcap\{\mathcal{F}(\kappa, \gamma) \mid \alpha \le \gamma \le \alpha + \kappa^+\}$. Suppose that $E_1 \subseteq E$.

Lemma 3.6: *Suppose that $C \in N_\alpha[G_{\kappa+1}, H_{\kappa+1}]$ is a $V[G_\kappa, H_\kappa]$-generic club through E defined as in Lemma 3.5. Then there exists $C_1 \in N_\alpha[G_{\kappa+1}, H_{\kappa+1}]$ a $V[G_\kappa, H_\kappa, C]$-generic club through E_1.*

Proof: Let $\langle \gamma_n \mid n < \omega \rangle$, $\langle \delta_n \mid n < \omega \rangle$ be the sequences used in the definition of C in Lemma 3.5.

For every $n < \omega$ there is $\gamma < \gamma_n$ such that $E_1(\kappa^+) \in \mathcal{F}(\kappa, \gamma)$. Then the set $\widetilde{E} = \{\delta < \kappa \mid \text{for some } \gamma < O^{\vec{\mathcal{F}}}(\delta) \quad E_1(\kappa^+) \cap \delta \in \mathcal{F}(\delta, \gamma)\}$ belongs to $\bigcap_{n<\omega} \mathcal{F}(\kappa, \gamma_n)$.

So the final segment of $\langle \delta_n \mid n < \omega \rangle$ is contained in $\widetilde{E}$. Suppose for simplicity that every δ_n is in $\widetilde{E}$.

Claim: *There exists a subsequence $\langle \delta_n^1 \mid n < \omega \rangle$ of a generic sequence to κ such that*

(1) $\bigcup_{n<\omega} \delta_n^1 = \kappa$,
(2) $\delta_n^1 < \delta_n$,
(3) *$C \cap \delta_n^1$ is a $V[G_{\delta_n^1}, H_{\delta_n^1}]$-generic club through $E \cap \delta_n^1$,*
(4) *in $V[G_{\delta_n+1}, H_{\delta_n+1}]$ there is a $V[G_{\delta_n^1}, H_{\delta_n^1}, C \cap \delta_n^1]$-generic club through $E_1 \cap \delta_n^1$.*

Proof: Let $\underset{\sim}{C}$ be a Q_κ-name of C over $N_\alpha[G_{\kappa+1}]$. Let $\langle q_n \mid n < \omega\rangle$ be a condition in Q_κ such that $q_n \in Q_{\delta_n+1}$. Denote by p_n the part of q_n in $Q_{\delta_n+1} - Q_{\delta_n}$. Thus $p_n \in V[G_{\delta_n+1}, Q_{\delta_n}]$ is a function from a subset of δ_n of a cardinality less than δ_n to $\{0,1\}$. Actually only p_n's are used in the definition of C in Lemma 3.5.

Since $\delta_n \in \widetilde{E}$, $E_1(\kappa^+) \cap \delta_n \in \mathcal{F}(\delta_n, \gamma)$ for some $\gamma < O^{\vec{\mathcal{F}}}(\delta_n)$. Then there is an element δ_n^1 of the generic sequence of δ_n such that $\sup(p_n) < \delta_n^1 < \delta_n$ and $\delta_n^1 \in E_1(\kappa^+)$. As in Lemma 3.5, then there exist C_n and C_n^1 so that C_n is a $V[G_{\delta_n^1}, H_{\delta_n^1}]$-generic club through $E \cap \delta_n^1$ and C_n^1 is a $V[G_{\delta_n^1}, H_{\delta_n^1}, c_n]$-generic club through $E_1 \cap \delta_n^1$. Extend now p_n to p_n' which forces "$\check{c}_n \subseteq \underset{\sim}{C}$". Let q' be the condition obtained by replacement of all p_n's by p_n''s. Then q' forces the statement of the claim. Since q was arbitrary, the empty condition forces the same. ∎

Let $\langle \delta_n^1 \mid n < \omega\rangle$ be as in the claim. Let C_n^1 be a $V[G_{\delta_n^1}, H_{\delta_n^1}, C \cap \delta_n^1]$ generic club through $E_1 \cap \delta_n^1$, for every $n < \omega$. Assume also that C_{n+1}^1 is an end extension of C_n^1. Set $C_1 = \bigcup C_n^1$. Let us check that C_1 is a $V[G_\kappa, H_\kappa, C]$-generic club through E_1.

Suppose that $\underset{\sim}{D} \in V[G_\kappa, H_\kappa]$ is a $P[E]$-name of a dense subset of $P[E_1]$. The set of ξ's such that $\underset{\sim}{D} \cap P[E_1 \cap \xi] \in V[G_\xi, H_\xi]$ and $\underset{\sim}{D} \cap P[E_1 \cap \xi]$ is forced by the empty condition to be a dense subset of $P[E_1 \cap \xi]$ containing a final segment of δ_n^1's. But $C \cap \delta_n^1$ is generic. So $D \cap P[E_1 \cap \delta_n^1]$ is dense in $V[G_{\delta_n^1}, H_{\delta_n^1}, C \cap \delta_n^1]$. Then C_1 is an end extension of an element of D. ∎

The rest of the construction is as in [G2].

In order to obtain a model with $\mathrm{NS}_\kappa^{\mathrm{Sing}}$-precipitous for inaccessible κ, use the construction above. Only (ω, κ^+)-repeat point is needed here since there is no need to care about regular limit points of the clubs.

Let now λ be a regular cardinal below κ. First use the forcing $\mathcal{P}_\kappa$. Then collapse κ to λ^+ by the Levy collapse. Now it is possible to force generic clubs. The Levy collapse is used to pick such clubs replacing the forcing Q_κ used above.

4. Presaturatedness of NS_κ

In this section we will give a rather sketchy construction of a model of ZFC+GCH with NS_κ presaturated for an inaccessible κ. Only new points which do not appear in the constructions of precipitous ideals will be emphasized. For the rest we refer to [J-M-M-P] and [G2]. Recall that an ideal over κ is presaturated if it is precipitous and the forcing with it preserves all the cardinals except perhaps κ itself.

Suppose that α^* is the least up-repeat point at κ in $\mathcal{K}(\vec{\mathcal{F}}) = V$. Let $\bar{\alpha}$ be the least ordinal $\leq O^{\vec{\mathcal{F}}}(\kappa)$ such that for every $A \in \mathcal{F}(\kappa, \alpha^*)$ there is β, $\alpha^* < \beta < \bar{\alpha}$ and $A \in \mathcal{F}(\kappa, \beta)$. Suppose for simplicity that $\bar{\alpha} = O^{\vec{\mathcal{F}}}(\kappa)$.

4.1 THE PREPARATION FORCING. Define first the iteration $\mathcal{P}_\alpha$ for α in the closure of $\{\beta \leq \kappa \mid \beta$ is an inaccessible or $\beta = \gamma + 1$ and γ is an inaccessible$\}$.

On the limit stages take the limit defined in [G2]. Suppose that α is an inaccessible and $\mathcal{P}_\alpha$ is defined. Define $\mathcal{P}_{\alpha+1}$. Let $C(\alpha^+)$ be the forcing for adding α^+-Cohen subsets to α, i.e., $\{f \in V^{\mathcal{P}_\alpha} \mid f$ is a partial function from $\alpha^+ \times \alpha$ into α, $|f|^{V^{\mathcal{P}_\alpha}} < \alpha\}$. $\mathcal{P}_{\alpha+1}$ will be $\mathcal{P}_\alpha * C(\alpha^+) * \mathcal{P}(\alpha, O^{\vec{\mathcal{F}}}(\alpha))$, where $\mathcal{P}(\alpha, O^{\vec{\mathcal{F}}}(\alpha))$ is a forcing for changing cofinality which is slightly different from those of [G2,3] used in section 3. We refer to [G2,3] for the detailed inductive definition and the properties. Let us just describe the changes we need to make here.

Define $U(\alpha, \gamma, t)$ which will be the ultrafilter extending $\mathcal{F}(\alpha, \gamma)$ for $\gamma < O^{\vec{\mathcal{F}}}(\alpha)$ and coherent sequence t. Let $j^\alpha_\beta \colon V \to N^\alpha_\beta \simeq V^\alpha / \mathcal{F}(\alpha, \beta)$ for $\beta < O^{\vec{\mathcal{F}}}(\alpha)$. Pick some well ordering W of V_λ for a big enough λ so that for every inaccessible $\delta < \lambda$, $W|\, V_\delta \colon V_\delta \leftrightarrow \delta$. Let γ be some fixed ordinal below $O^{\vec{\mathcal{F}}}(\alpha)$. Drop for a while the indexes α, γ in j^α_γ, N^α_γ.

Let $\langle \underset{\sim}{A}_{\gamma'} \mid \gamma' < \alpha^+ \rangle$ be the $j^\alpha_0(W)$-least enumeration of all canonical $\mathcal{P}_\alpha * C(\alpha^+)$-names of subsets of α.

Over $N^{j(\mathcal{P}_\alpha)}$ define an increasing sequence $\langle \underset{\sim}{y}_{\gamma'} \mid \gamma' < \alpha^+ \rangle$ of conditions in $j(C(\alpha^+))$ deciding the statement "$\check{\beta} \in j(\underset{\sim}{A}_{\gamma'})$" ($\beta \leq \alpha$, $\gamma' < \alpha^+$). Let us pick in the definition always $j^\alpha_0(W)$-least extension. Let us now make changes in the sequence $\langle y_{\gamma'} \mid \gamma' < \alpha^+ \rangle$.

Define a sequence $\langle \underset{\sim}{p}_{\gamma'} \mid \gamma' < \alpha^+ \rangle$ of elements of $\mathcal{P}_{j(\alpha)} / \mathcal{P}_{\alpha+1}$ so that

(1) for a limit $\gamma' < \alpha^+$ $\| \underset{\sim}{p}_{\gamma'}$ is the $j(W)$-least Easton extension of $\langle \underset{\sim}{p}_{\gamma''} \mid \gamma'' < \gamma' \rangle \|^{\mathcal{P}_{\alpha+1}} = 1$,

(2) for every $\gamma' < \alpha^+$ $\| \underset{\sim}{p}_{\gamma'+1}$ is the $j(W)$-least Easton extension of $\underset{\sim}{p}_{\gamma'}$ deciding "$\underset{\sim}{y}_{\gamma'} \Vdash \check{\alpha} \in j(\underset{\sim}{A}_{\gamma'})$" $\|^{\mathcal{P}_{\alpha+1}} = 1$.

For a subset A of α set $A \in U(\alpha, \gamma, t)$ if for some r in the generic subset of $\mathcal{P}_\alpha * C(\alpha^+)$, some $\gamma' < \alpha^+$, a name $\underset{\sim}{A}$ of A and a $\mathcal{P}_\alpha * C(\alpha^+)$ name $\underset{\sim}{T}$, in N

$$r \cup \{\langle \check{t}, \underset{\sim}{T} \rangle\} \cup p_{\gamma'} \cup y_{\gamma'} \Vdash (\check{\xi}, \check{\alpha}) \in j(\underset{\sim}{A}) \,.$$

Let us consider now the ultrafilter $U(\alpha,\gamma,t)\times U(\alpha,\delta,\phi)$ for some t and δ. Let

$$i\colon V \longrightarrow M \simeq V^{\kappa^2}/\mathcal{F}(\alpha,\gamma)\times\mathcal{F}(\alpha,\delta)\ .$$

Then $A\in U(\alpha,\gamma,t)\times U(\alpha,\delta,\phi)$ iff $\{\xi \mid \{\mu \mid (\xi,\mu)\in A\}\in U(\alpha,\delta,\phi)\}\in U(\alpha,\gamma,t)$ iff $\{\xi \mid$ for some r in the generic subset of $\mathcal{P}_\alpha * C(\alpha^+)$, some $\gamma'<\alpha^+$, a name $\underset{\sim}{A}$ of A and a $\mathcal{P}_\alpha * C(\alpha^+)$ name $\underset{\sim}{T}$, in N^α_δ, $r\cup\{\langle\phi,\underset{\sim}{T}\rangle\}\cup p_{\gamma'}\cup y_{\gamma'} \Vdash ((\check{\xi},\check{\alpha}))\in j^\alpha_\delta(\underset{\sim}{A})\}\in \mathcal{U}(\alpha,\gamma,t)$.

Fix some name $\underset{\sim}{A}$ of A. For each $\xi<\alpha$ pick a maximal antichain $\{r^\rho_\xi \mid \rho\in I_\xi\}$ of elements of $\mathcal{P}_\alpha * C(\alpha^+)$ so that each r^ρ_ξ decides in N^α_δ the statement "for some $\gamma'_\xi<\alpha^+$, some $\underset{\sim}{T}$ $\langle\phi,\underset{\sim}{T}\rangle\cup p_{\gamma'_\xi}\cup y_{\gamma'_\xi}|.\ \Vdash(\check{\xi},\check{\alpha})\in j^\alpha_\delta(\underset{\sim}{A})$". Since $\mathcal{P}_\alpha * C(\alpha^+)$ satisfies α^+-c.c., γ'_ξ's can be fixed. Denote $\bigcup\{\gamma'_\xi \mid \xi<\alpha\}$ by $\gamma(\alpha)$. Then, $A\in U(\alpha,\gamma,t)\times U(\alpha,\delta,\phi)$ iff there exist r_1 in the generic subset of $\mathcal{P}_\alpha * C(\alpha^+)$, $\gamma_1<\alpha^+$ and a $\mathcal{P}_\alpha * C(\alpha^+)$ name $\underset{\sim}{T}_1$ so that in N, $r_1\cup\{\langle\check{t},\underset{\sim}{T}_1\rangle\}\cup p_{\gamma_1}\cup y_{\gamma_1}\Vdash$ (for some $\rho\in j(I_\alpha)$ s.t. r^ρ_α is in the generic subset of $\mathcal{P}_{j(\alpha)} * C\big(j(\alpha^+)\big)$ r^ρ_α forces in M "for some $\underset{\sim}{T}$, $\langle\phi,\underset{\sim}{T}\rangle\cup p_{j(\gamma(\alpha))}\cup y_{j(\gamma(\alpha))}\Vdash(\check{\alpha},j(\check{\alpha}))\in j(\underset{\sim}{A})$").

Note that it is possible to replace $\gamma(\alpha)$ by γ_1.

Consider now $U(\alpha,\gamma,t)\times U(\alpha,\gamma,\phi)$. Let $k\colon V[G_\alpha,H_\alpha]\to N^*[G_{k(\alpha)},H_{k(\alpha)}]$ be the corresponding elementary embedding, where $G_\alpha * H_\alpha$ is a generic subset of $\mathcal{P}_\alpha * C(\alpha^+)$. Let us change one value of the function $H_{k(\alpha)}(0)$ so that it will take α to $[id_2]$, where $id_2(\tau,\nu)=\nu$. Fix some $h\colon \alpha^+\leftrightarrow\gamma$, $h\in V$. For $\delta<\gamma$ let $\pi_{\gamma\delta}$ be a projection of $U(\alpha,\gamma,\phi)$ onto $U(\alpha,\delta,\phi)$. For every $\delta<\alpha^+$ change one value of $H_{k(\alpha)}\big(k(\delta)\big)$ so that it will take α to $[\pi_{\gamma h(\delta)}\circ id_2]$.

Denote by $\bar{H}_{k(\alpha)}$ the changed $H_{k(\alpha)}$. By the arguments of Woodin see [Sh-W], $\bar{H}_{k(\alpha)}$ is still generic. Let $\bar{k}\colon V[G_\alpha,H_\alpha]\to N^*[G_{k(\alpha)},\bar{H}_{k(\alpha)}]$ be the appropriate elementary embedding. Define an ultrafilter $\bar{U}(\alpha,\gamma,t)$ as follows:

$$A\in\bar{U}(\alpha,\gamma,t)\quad\text{iff}\quad \alpha\in\bar{k}(A)\ .$$

Use $\langle\bar{U}(\alpha,\gamma,t)\mid\gamma<O^{\vec{\mathcal{F}}}(\alpha),\ t \text{ a coherent sequence}\rangle$ in the forcing $\mathcal{P}\big(\alpha,O^{\vec{\mathcal{F}}}(\alpha)\big)$ instead of just $\langle U(\alpha,\gamma,t)\mid\gamma<O^{\vec{\mathcal{F}}}(\alpha),\ t \text{ a coherent sequence}\rangle$ used in [G2].

Set $\mathcal{P}_{\alpha+1}=\mathcal{P}_\alpha * C(\alpha^+) * \mathcal{P}\big(\alpha,O^{\vec{\mathcal{F}}}(\alpha)\big)$.

It completes the inductive definition of the iteration $\mathcal{P}_\alpha$ $(\alpha\le\kappa)$.

Over $V^{\mathcal{P}_\kappa * C(\kappa^+)}$ force with the forcing Q_κ defined in section 3. Let $G_\kappa * H_\kappa * F_\kappa$ be a generic subset of $\mathcal{P}_\kappa * C(\kappa^+) * Q_\kappa$. The model $V[G_\kappa * H_\kappa * F_\kappa]$ is the desired preparation model.

4.2. THE MAIN FORCING. As in section 3 we would like to shoot a lot of clubs in order to make NS_κ presaturated. But now we need to destroy much more stationary sets. Analyzing the forcing with NS_κ of section 3 it is not hard to see that κ, κ^+ are collapsed to ω. One of the reasons (there are many others) is that the forcing for shooting a club in the ultrapower N for $j(\kappa)$ actually shoots a club into a nonstationary set. Since cf $\big(j(\kappa)\big) = \kappa^+$ (in $K(\vec{\mathcal{F}})$) and we are shooting clubs into sets A consisting of regular in $\mathcal{K}(\vec{\mathcal{F}})$ cardinals, so $j(A)$ consists of ordinals of cofinality κ^+. But $\{\beta < j(\kappa) \mid \text{cf } \beta = \kappa^+\}$ is nonstationary in V.

In order to eliminate this situation let us insure from the beginning that $j(\kappa)$ (for a generic j) will be a limit of indiscernibles which are almost contained in every $j(A)$ whenever a club should be shot through A.

By [G5], $\langle U(\kappa, \gamma, \phi) \mid \gamma < O(\kappa)\rangle$ forms a Rudin–Kiesler increasing sequence in $V[G_\kappa * H_\kappa]$. Let us pick $\langle U(\kappa, \gamma, \phi) \mid \gamma < \alpha^*\rangle$ and form a direct limit of the ultrapowers, where α^* is an up-repeat point. It is well founded and closed under κ-sequences since cf $\alpha^* = \kappa^+$. Using the method of [G5] we'll turn this direct limit into usual ultrapower. But first let us define filters $\langle W(\kappa, \beta) \mid \beta < O(\kappa)\rangle$. Set $A \in W(\kappa, \beta)$ iff there exist $r \in G_\kappa * H_\kappa * F_\kappa$ and $\gamma < \kappa^+$ so that in N^κ_β, $r \cup 1_{\mathcal{P}(\kappa,\beta)} \cup p_\gamma \cup y_\gamma \Vdash \check{\kappa} \in j^\kappa_\beta(\underset{\sim}{A})$.

It is easy to see that every $W(\kappa, \beta)$ is precipitous and the forcing with it is isomorphic to $\mathcal{P}(\kappa, \beta)$ followed by $Q_{j(\kappa)}/Q_\kappa$.

Fix some $\beta \geq \alpha^*$. Consider the filters $W(\kappa, \beta) \times U(\kappa, \gamma, \phi)$ for $\gamma \leq \beta$.

They still form a Rudin–Kiesler increasing sequence. We are interested in $W(\kappa, \beta)\times$ the direct limit of $\langle U(\kappa, \gamma, \phi) \mid \gamma < \alpha^*\rangle)$, since $j(\kappa)$ in the ultrapower will be a limit of indiscernibles. Let us use the generic functions H_κ in order to turn $W(\kappa, \beta)\times$ direct limit of $\langle U(\kappa, \gamma, \phi) \mid \gamma < \alpha^*\rangle$ into a normal filter.

Denote by i^β_γ: $N^\kappa_\beta \to M^\beta_\gamma \approx (N^\kappa_\beta)^{j^\kappa_\beta(\kappa)}/j^\kappa_\beta(\mathcal{U}(\kappa, \gamma, \emptyset)$. Then, as in 4.1, $A \in W(\kappa, \beta) \times U(\kappa, \gamma, \phi)$ if there are $r_1 \in G_\kappa * H_\kappa * F_\kappa$ and $\gamma_1 < \kappa^+$ so that in N^κ_β, $r_1 \cup 1_{\mathcal{P}(\kappa,\beta)} \cup p_{\gamma_1} \cup y_{\gamma_1} \Vdash$ for some $\rho \in j^\kappa_\beta(I_\kappa)$ s.t. r^ρ_κ is in the generic subset of $\mathcal{P}_{j^\beta_\kappa(\kappa)} * C\big(j^\beta_\kappa(\kappa^+)\big)$ r^ρ_κ forces in M^β_γ "for some $\underset{\sim}{T}$ $\langle \phi, \underset{\sim}{T}\rangle \cup p_{j(\gamma_1)} \cup y_{j(\gamma_1)} \Vdash\!\!\!\!\!— (\check{\kappa}, j^\kappa_\beta(\check{\kappa})) \in i^\beta_\gamma(\underset{\sim}{A})$.

Let us change values as in 4.1 but only for the functions with indexes in $h^{-1\prime\prime}(\gamma)$ (where h: $\kappa^+ \longleftrightarrow \gamma$). It is equivalent to change of $y_{\gamma'}$'s. Namely, if $h(\delta) < \gamma$ and $i_\gamma(\delta) \in \text{dom } y_{j(\gamma')}$ $(\gamma' < \kappa^+)$, then change $y_{j(\gamma')}\big(i_\gamma(\delta)\big)$ to $i_\gamma(\pi_{\gamma h(\delta)} \circ id_2)\big(\kappa, j^\kappa_\beta(\kappa)\big) = i_\gamma(\pi_{\gamma h(\delta)})\big(j^\kappa_\beta(\kappa)\big)$. Let us denote the changed $y_{\gamma'}$ by the same letter.

We are not interested in all the information $y_{j(\gamma')}$'s give; the important part is $y_{j(\gamma')}| h^{-1''}(\gamma)$.

Define the set $W^*(\kappa,\beta)$ as follows:

$A \in W^*(\kappa,\beta)$ if for some $\gamma < \alpha^*$ there are $r_1 \in G_\kappa * H_\kappa * F_\kappa$ and $\gamma_1 < \kappa^+$ so that in N^κ_β, $r_1 \cup 1_{\mathcal{P}(\kappa,\beta)} \cup p_{\gamma_1} \cup y_{\gamma_1} \Vdash$ "for some $\rho \in j^\kappa_\beta(I_\kappa)$ s.t. r^ρ_κ is in the generic subset of $\mathcal{P}_{j^\beta_\kappa(\kappa)} * C(j^\beta_\kappa(\kappa^+))$ r^ρ_κ forces in M^β_γ "for some $\underset{\sim}{T}$ $\langle \phi, \underset{\sim}{T}\rangle \cup p_{j(\gamma_1)} \cup y_{j(\gamma_1)}|h''^{-1}(\gamma) \Vdash \check{\kappa} \in i^\beta_\gamma(\underset{\sim}{A})$".

LEMMA 4.1: *$W^*(\kappa,\beta)$ is a filter in $V[G_\kappa * H_\kappa * F_\kappa]$.*

Proof: The following is sufficient.

CLAIM: *Let $A \subseteq \kappa$. If for some $\gamma < \alpha^*$ there are $r_1 \in G_\kappa * H_\kappa * F_\kappa$ and $\gamma_1 < \kappa^+$ so that in N^β_κ, $r_1 \cup 1_{\mathcal{P}(\kappa,\beta)} \cup p_{\gamma_1} \cup y_{\gamma_1} \Vdash$ "for some $\rho \in j^\kappa_\beta(I_\kappa)$ s.t. r^ρ_κ is in the generic subset $\mathcal{P}_{j^\beta_\kappa(\kappa)} * C(j^\beta_\kappa(\kappa^+))$, r^ρ_κ forces in M^β_γ "for some $\underset{\sim}{T}$*

$$\langle \phi, \underset{\sim}{T}\rangle \cup p_{j^\kappa_\beta(\gamma_1)} \cup y_{j^\kappa_\beta(\gamma_1)}|h''^{-1}(\gamma) \Vdash \check{\kappa} \in i_\gamma(\underset{\sim}{A})$$

then for every δ, $\gamma < \delta < \alpha^*$ the same statement holds with γ replaced by δ.

Proof: Consider the set $B = \{(\nu,\eta) \in [\kappa]^2 \mid$ for some $r \in (G_\kappa * H_\kappa * F_\kappa)| \eta$, some $\underset{\sim}{T}$ s.t. $r \Vdash$ "$\langle \phi, \underset{\sim}{T}\rangle \in \mathcal{P}(\eta, O(\eta))$" and $p_{\gamma_1}, y_{\gamma_1}$ defined using the well ordering until the step γ_1, $r \bigcup \{\langle \phi, \underset{\sim}{T}\rangle\} \cup p_{\gamma_1} \cup y_{\gamma_1}|h''^{-1}(\gamma) \Vdash \check{\nu} \in \underset{\sim}{A}\}$.

Extend $W^*(\kappa,\beta)$ to κ-complete ultrafilters W_1,W_2 in a reasonable fashion fixing respectively γ and δ. Then $B \in W_1$. Using $W_1 \leq_{RK} W_2$, the image of B in the ultrapower with W_2 will have the same description as in the ultrapower with W_1. Hence $B \in W_2$. Now use the definitions of B and W_2. ∎

LEMMA 4.2: *$W^*(\kappa,\beta)$ is a normal filter.*

The proof is routine.

It is not hard to see that the forcing with $W^*(\kappa,\beta)$ is isomorphic to $\mathcal{P}(\kappa,\beta)*$ (a part of $C(j(\kappa^+)) * (Q_{j(\kappa)}/Q_\kappa)$), where $j(\kappa)$ is the same as the image of κ under $U(\kappa,\beta,\phi)$ followed by the direct limit of $\langle U(\kappa,\gamma,\phi) \mid \gamma \in \alpha^*\rangle$. Since the forcing above preserves all the cardinals, $W^*(\kappa,\beta)$ is presaturated.

Now we have to shoot clubs through sets in $W(\kappa,\beta)$ for a lot of β's and iterate this process. Why is it possible without collapsing cardinals?

Consider for a moment ultrapower with $U(\kappa,\beta,\phi) \times U(\kappa,\alpha^*,\phi)$. The forcing which is used over $j(\kappa)$ there is based on the image of $\langle U(\kappa,\gamma,t) \mid \gamma < \alpha^*\rangle$.

The candidates for shooting clubs will always be in an intersection of a closed unbounded part of $U(\kappa,\gamma,\phi)$'s or their extensions. So this forcing will actually be closed from outside.

LEMMA 4.3: *α^* is an up-repeat point for $\langle U(\kappa,\gamma,\phi) \mid \gamma < O(\kappa)\rangle$.*

Proof: Let $A \in U(\kappa,\alpha^*,\phi)$. Find some $r \in G_\kappa * H_\kappa$, $\gamma' < \kappa^+$, $\underset{\sim}{T}$, $\underset{\sim}{A}$ so that in $N^\kappa_{\alpha^*}$, $r\bigcup\{\langle\phi,\underset{\sim}{T}\rangle\} \cup p_{\gamma'} \cup y_{\gamma'} \Vdash$ "$\check{\kappa} \in j^{\alpha^*}_\kappa(\underset{\sim}{A})$".

Consider the set $B = \{\alpha < \kappa \mid r|\ \alpha \cup \langle\phi,\underset{\sim}{T}|\alpha\rangle \cup$ "$\check{\alpha} \in \underset{\sim}{A}$"$\}$, where $r|\ \alpha$, $\underset{\sim}{T} \mid \alpha$ are the natural restrictions to α and $p^\alpha_{\gamma'}$, $y^\alpha_{\gamma'}$ are defined over α in the same fashion as $p_{\gamma'}$, $y_{\gamma'}$ were defined over κ.

Clearly, $B \in \mathcal{F}(\kappa,\alpha^*)$. So for some β, $\alpha^* < \beta < O(\kappa)$, $B \in \mathcal{F}(\kappa,\beta)$. But then in N^κ_β, $r\bigcup\{\langle\phi,\underset{\sim}{T}\rangle\} \cup p^\beta_{\gamma'} \cup y^\beta_{\gamma'} \Vdash$ "$\check{\kappa} \in j^\beta_\kappa(\underset{\sim}{A})$". It means that $A \in U(\kappa,\beta,\phi)$. ∎

Notice that $\langle U(\kappa,\gamma,\phi) \mid \gamma < O(\kappa)\rangle$ is not a coherent sequence. So the next lemma does not follow directly from section 1.

LEMMA 4.4: *If $A \in \bigcap_{\alpha^*<\beta<O(\kappa)} U(\kappa,\beta,\phi)$, then for some $\tau < \alpha^*$*

$$A \in \bigcap_{\tau<\beta\le\alpha^*} U(\kappa,\beta,\phi).$$

Proof: Suppose otherwise. Consider the set $B = \{\alpha < \kappa \mid$ for every $\delta < \kappa$ some member of the generic sequence to α above δ is not in $A\}$.

Then $B \in U(\kappa,\alpha^*,\phi)$. By Lemma 4.3, $B \in U(\kappa,\beta,\phi)$, for some $\beta > \alpha^*$. But then $A \notin U(\kappa,\beta',\phi)$ for unboundedly many $\beta' < \beta$, since $\langle U(\kappa,\gamma,\phi) \mid \gamma < O(\kappa)\rangle$ forms a Rudin–Kiesler increasing sequence with projections on the generic sequences. See [G5] for more details. ∎

Now for which β's will $W^*(\kappa,\beta)$ be used? It is impossible just to use $\bigcap\{W^*(\kappa,\beta) \mid \alpha^* \le \beta\}$ since we need for precipitousness a separating family, i.e., a set $\langle A_\beta \mid \beta \ge \alpha^*\rangle$ s.t. $A_\beta \in W(\kappa,\beta)$ and does not belong to the rest of the filters. But already the sequence $\langle \mathcal{F}(\kappa,\beta) \mid \beta \ge \alpha^*\rangle$ contains weak repeat points. Let us remove all redundant β's.

Let $\langle A_\nu \mid \nu < \kappa^+\rangle$ be an enumeration of a generating family of $\mathcal{F}(\kappa,\alpha^*)$ so that $|A_{\nu+1}-A_\nu| < \kappa$. Define by induction sequences $\langle \tau_\nu \mid \nu < \kappa^+\rangle$, $\langle \beta_\nu \mid \nu < \kappa^+\rangle$, $\alpha^* < \beta_\nu < O(\kappa)$. Let β_1 be the least ordinal above α^* so that $A_0 \in \mathcal{F}(\kappa,\beta)$. Set $\tau_0 = 0$. Let τ_1 be the least successor ordinal ν so that $A_\nu \notin \mathcal{F}(\kappa,\beta_1)$. If

$\nu = \nu' + 1$, then let β_ν be the least ordinal β above $\beta_{\nu'}$ so that $A_{\tau_{\nu'}} \in \mathcal{F}(\kappa, \beta)$. Let τ_ν be the least successor ordinal γ s.t. $A_\gamma \notin \mathcal{F}(\kappa, \beta_\nu)$.

For a limit ν, pick β_ν to be the least ordinal $\beta > \bigcup_{\nu'<\nu} \beta_{\nu'}$ so that $A_{\bigcup_{\nu'<\nu} \tau_{\nu'}} \in \mathcal{F}(\kappa, \beta)$. Define τ_ν as on the successor stage.

Let $B = \{\beta_\nu \mid \nu < \kappa^+\}$. Then for every $A \in \mathcal{F}(\kappa, \alpha^*)$ there is $\beta \in B$ so that $A \in \mathcal{F}(\kappa, \beta)$. Also $\langle \kappa - A_{\tau_\nu} \mid \nu < \kappa^+ \rangle$ is a separating family for $\langle \mathcal{F}(\kappa, \beta_\nu) \mid \nu < \kappa^+ \rangle$.

We would like to add to B more ordinals, preserving the separation property, namely for every $\beta \in B$ a closed unbounded subsequence of $\square_{\kappa^+}$ sequence to β. Let $\langle C^\kappa_\alpha \mid \alpha < \kappa^{++} \rangle$ be the $\square_{\kappa^+}$-box sequence. By Section 1, cf $\alpha^* = \kappa^+$. So every $A \in \mathcal{F}(\kappa, \alpha^*)$ belongs to unboundedly many measures in δ, for a closed unbounded set of δ's in $C^\kappa_{\alpha^*}$. Let us assume that $\langle A_\nu \mid \nu < \kappa^+ \rangle$ was picked so that for every $\nu < \kappa^+$, $\{\beta < \kappa \mid A_\nu \cap \beta$ belongs to unboundedly many in δ measures for a club of δ's in $C^\beta_{O^{\mathcal{F}}(\beta)}\} \supseteq A_{\nu+1}$.

Now add to B for every $\nu < \kappa^+$ a club of β's in $C^\kappa_{\beta_\nu}$ s.t.

(1) $\beta > \max(\bigcup_{\nu'<\nu} \beta_{\nu'}, \alpha^*)$,

(2) there exists a maximal $\gamma < \beta$ so that $A_{\bigcup_{\nu'<\nu} \tau_{\nu'}}$ belongs to $\mathcal{F}(\kappa, \gamma)$.

Denote the set obtained so by B^*. By the construction, the family $\langle \mathcal{F}(\kappa, \gamma) \mid \gamma \in B^* \rangle$ can be separated.

Applying arguments of section 1, it is not hard to see that if $A \in \bigcap\{\mathcal{F}(\kappa, \gamma) \mid \gamma \in B^*\}$, then there is a club $C \subseteq C^\kappa_{\alpha^*}$ so that $A \in \bigcap\{\mathcal{F}(\kappa, \gamma) \mid \gamma = \alpha^*$ or $\gamma \in C\}$.

Let us use the filters $\bigcap\{\mathcal{F}(\kappa, \beta) \mid \beta \in B^*\}$ and $\bigcap\{W^*(\kappa, \beta) \mid \beta \in B^*\}$ as a basic one for shooting clubs. The forcing applied here is analogous to those used in section 3 and in [G2]. One new problem appears in the present context: which projection of the forcings $\mathcal{P}(\kappa, \beta) * Q_{\kappa+1}/Q_\kappa$ onto the forcings for shooting clubs to pick. It was not important for constructing precipitous ideals, since cardinals there may collapse. And this actually happens if the projection is picked in a generic way as in section 3 or [J-M-M-P] or [G2]. The simplest thing to do is just to fix some projection from the beginning. The problem is that this adds new stationary sets into the filters and it is unclear how to turn them into clubs without collapsing cardinals. So let us do something in the middle. We will turn the forcing for choosing the projection into an atomic one.

Denote by B_α all the elements $\beta \in B^*$ so that otp$C^\kappa_\beta = \alpha$, for every $\alpha \leq \kappa^+$. Let $\langle R_\alpha \mid \alpha \leq \kappa^+ \rangle$ denote the iterated forcing for shooting clubs over $V[G_\kappa * H_\kappa * F_\kappa]$. Let $\alpha < \kappa^+$ and $\beta \in B_\alpha$. We project $\mathcal{P}(\kappa, \beta)$ onto R_α free. But for $\gamma > \alpha$ fix some canonical projection of $\mathcal{P}(\kappa,\beta)/R_\alpha$ onto R_γ. Note that on each stage γ of the iteration $\langle R_\xi \mid \xi < \kappa^+\rangle$ only projections for β's in B^* with $\mathrm{otp}C^\kappa_\beta < \gamma$ are fixed. But such elements appear only on an initial segment of the Radin clubs. So it still leaves enough freedom for shooting clubs.

5. What can happen over a measurable?

Let us start from saturation. By Namba [N], NS_κ or even $\mathrm{NS}^{\mathrm{Reg}}_\kappa = \mathrm{NS}\,|$ (regular cardinals) cannot be saturated over a measurable κ. Consider the following weaker notion.

Definition 5.1: An ideal I is densely saturated if for every I-positive set A there exists an I-positive set $B \subseteq A$ so that $I|\,B$ is saturated.

Clearly, saturation implies dense saturation and dense saturation implies presaturation.

We do not know whether full nonstationary ideal or $\mathrm{NS}^{\mathrm{Sing}}_\kappa$ can be densely saturated. But $\mathrm{NS}^{\mathrm{Reg}}_\kappa$ can be even saturated; Jech and Woodin [J-W] have shown the consistency of "NS^{Reg}_κ saturated" follows from that of a measurable cardinal.

THEOREM 5.1: *"$\mathrm{NS}^{\mathrm{Reg}}_\kappa$ is densely saturated over a measurable κ" is equiconsistent with a weak repeat point.*

Proof: If $\mathrm{NS}^{\mathrm{Reg}}_\kappa$ is densely saturated over a measurable κ, then it is precipitous. But by [G6] it implies a weak repeat point over κ.

On the other hand, if there exists a weak repeat point over κ in $\mathcal{K}(\vec{\mathcal{F}})$, then use the construction of Jech-Woodin [J-W], only each stage $\alpha \le \kappa$ of interaction shoot clubs into sets in $\bigcap\{\mathcal{F}(\alpha,\beta) \mid \beta < O(\alpha)\}$ and the filters generated by this filter. ∎

Turn now to presaturation and precipitousness.

Definition 5.2: Let us say that $\beta < O^{\vec{\mathcal{F}}}(\kappa)$ is a weak repeat point above α if for every $A \in \mathcal{F}(\kappa,\beta)$ there is γ, $\alpha \le \gamma < \beta$ so that $A \in \mathcal{F}(\kappa,\gamma)$.

THEOREM 5.3: *Suppose that there is no inner model of $\exists\kappa O(\kappa) = \kappa^{++}$. Then the following holds.*

1. *If $\mathrm{NS}^\lambda_\kappa$ is precipitous (or presaturated) over a measurable κ for some $\lambda < \kappa$ then, in $\mathcal{K}(\vec{\mathcal{F}})$, there exists a measure concentrating over $(\omega, <\kappa)$-repeat points (or up-repeat points respectively).*

2. *If* NS_κ *is precipitous (or presaturated) over a measurable* κ *then, in* $\mathcal{K}(\vec{\mathcal{F}})$, *there exists an* $(\omega, <\kappa)$*-repeat point (or an up-repeat point) with a weak repeat point above it.*

Proof: For (1) just note that $\mathrm{NS}_\kappa^\lambda$ will be still precipitous or presaturated in the ultrapower. Let us prove (2). Let $j\colon V \to M$ be the ultrapower of V by a normal ultrafilter over κ. By [Mi1], $j| \mathcal{K}(\vec{\mathcal{F}})$ is an iterated ultrapower. Let $\mathcal{F}(\kappa,\beta)$ be a measure used to move κ. So $j(\vec{\mathcal{F}})|\,(\kappa+1) = \vec{\mathcal{F}}|\,\beta$. By section 2, there exists an $\alpha < \beta$ which is an $(\omega, <\kappa)$-repeat point if NS_κ is precipitous or up-repeat point if NS_κ is presaturated. If there exists a set $A \in \mathcal{F}(\kappa,\beta)$ which does not belong to any $\mathcal{F}(\kappa,\gamma)$ for $\alpha \le \gamma < \beta$, then $\mathcal{F}(\kappa,\beta)$ can be recovered in M from $\mathrm{NS}_\kappa \mid A$. ∎

THEOREM 5.4:

(1) *If there exists a measure over* κ *in* $\mathcal{K}(\vec{\mathcal{F}})$ *concentrating over* (ω,κ^+)*-repeat points (or up-repeat points), then in a generic extension* $\mathrm{NS}_\kappa^{\mathrm{Sing}}$ *is precipitous (or presaturated) over a measurable* κ.

(2) *If there exists an* $(\omega,\kappa^+ + 1)$*-repeat point (or an up-repeat point) with a weak repeat point above it, then in a generic extension* NS_κ *is precipitous (or presaturated) over a measurable* κ.

Proof: We deal with a precipitousness case. The presaturation case is similar, only the forcing of section 4 should be used.

(1) Let α be an (ω,κ^+)-repeat point for $\vec{\mathcal{F}}$ over κ . The hypothesis is equivalent to the assertion that there is an (ω,κ^+)-repeat point such that $\mathcal{F}(\kappa,\alpha+\kappa^+)$ exists. (Note that it need not exist for an arbitrary (ω,κ^+)-repeat point.) Assume that κ is the least δ so that $O^{\vec{\mathcal{F}}}(\delta) = \delta' + \delta + 1$ for an (ω,δ^+)-repeat point δ'. Let $A = \{\delta < \kappa \mid$ there exists an (ω,δ^+)-repeat point $\delta' < O^{\vec{\mathcal{F}}}(\delta)$ so that $O^{\vec{\mathcal{F}}}(\delta) = \delta' + \delta^+\}$. Then $A \in \mathcal{F}(\kappa, \alpha+\kappa^+)$. Define the iterated forcing notion as follows. At every stage $\delta \in (\kappa+1) - A$ use the forcing defined in section 3. For $\delta \in A$, use the forcing of section 3 for making NS_δ precipitous (i.e., the forcing used there over κ itself). Such iteration preserves the cardinals since each forcing used on stage δ for $\delta \in A$ is embeddible into Prikry type forcing notion. We refer to [G6] for this matter. In the final model $\mathcal{F}(\kappa,\alpha+\kappa^+)$ extends to a normal ultrafilter, since the forcing which should be used over κ in the ultrapower by $\mathcal{F}(\kappa,\alpha+\kappa^+)$ is the same as those used over κ in V.

(2) Assume that κ is picked minimal carrying a weak repeat point above an

$(\omega, \kappa^+ + 1)$-repeat point. Denote the $(\omega, \kappa^+ + 1)$ repeat point by α and a weak repeat point above it by β.

Define the iterated forcing notion as follows. For $\delta < \kappa$ without $(\omega, \delta^+ + 1)$-repeat point use the forcing of section 3. If $\delta \le \kappa$ has an $(\omega, \delta^+ + 1)$-repeat point δ', then use the forcing for shooting clubs of section 3 for the filter $\bigcap\{\mathcal{F}(\delta, \gamma) \mid \delta' \le \gamma < O^{\vec{\mathcal{F}}}(\delta)\}$ which clearly is contained in the filter $\bigcap\{\mathcal{F}(\delta, \gamma) \mid \delta' \le \gamma \le \delta' + \delta^+\}$ used in section 3.

The measure $\mathcal{F}(\kappa, \beta)$ extends in such generic extension since the filters $\bigcap\{\mathcal{F}(\kappa, \gamma) \mid \alpha \le \gamma < \beta\}$ and $\bigcap\{\mathcal{F}(\kappa, \gamma) \mid \alpha \le \gamma \le \beta\}$ are the same, and so the forcing used over κ in V is the forcing which should be used in the ultrapower by $\mathcal{F}(\kappa, \beta)$. ∎

Remarks:

1. The question whether NS_κ can be precipitous over a measurable was raised by Baumgartner–Taylor–Wagon [B-T-W]. Models with NS_κ precipitous over a measurable were contracted first by Foreman–Magidor–Shelah [F-M-S] from a super-strong above supercompact and later from a supercompact alone in [G6].
2. It is possible to do the above constructions over a supercompact. Thus the strength of "NS_κ is precipitous over a κ^+-supercompact cardinal κ" can be reduced to κ^+-supercompact alone.

6. Some open problems

1. Is the existence of $(\omega, <\kappa)$ or (ω, κ) repeat point sufficient for a model with $NS_\kappa^{\aleph_0}$ or NS_κ precipitous for an inaccessible κ?
2. Does the precipitousness of NS_κ over $\kappa \ge \aleph_3$ imply an $(\omega, \lambda + 1)$ repeat point where $\kappa = \lambda^+$?
3. Does the saturatedness of $NS_\kappa^{\aleph_0}$ over an inaccessible κ imply $\exists \alpha O(\alpha) = \alpha^{++}$ in an inner model?
4. Is it consistent "NS_κ saturated over an inaccessible κ"?
5. Is it consistent "$NS_\kappa^{\aleph_0}$ saturated over an inaccessible κ"?
6. Is "NS_κ precipitous and after the forcing with it κ remains uncountable" weaker than up-repeat point?
7. Is it consistent "NS_κ (or $NS_\kappa^{\aleph_0}$) is densely saturated"?

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